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Aspects of estimating inter subject variability with a constant intra coefficient of variation in repeated measurements

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Abstract

We study the one-way random effects model of Chow & Tse (1991) with homogeneous intra subject coefficient of variation and deal with the problem of estimation of the basic parameters. We propose and study properties of the inter variability along the lines of Mathew, Sinha & Sutradhar (1992). We also study properties of an appropriate Bayes estimator.

Key words: Coefficient of variation, variance components, linear model, multiplicative model, parameter estimation, Bayesian analysis, MCMC

AMS Subject Classification: primary: 62H12; secondary: 62H99

1 Introduction

In this paper an interesting repeated measurements model is investigated. It is of great use when modelling many biological processes where the variance is coupled to the mean, violating the standard assumption of a constant variance. To overcome some of the difficulties with a non-constant variance one often log-transforms data. However, instead of assuming a constant variance it is in many cases realistic to use a constant coefficient of variation and this is the basic ingredient in the present model. In the context of an assay validation or an instrument validation process, Chow & Tse (1991) and Yang & HayGlass (1993) proposed the model. One may classify the model as a one-way random effects model with homogeneous intra subject coefficient of variation. Denoting by y_{ij} the jth observation of the ith subject ($j = 1, \ldots, k_i$, $i = 1, \ldots, n$), the model is given by

$$y_{ij} = z_i + \varepsilon_{ij}, \tag{1}$$

where z's are independent normal with mean μ and variance σ^2, and conditionally given z's, ε's are independent normal with mean 0 and variance $\delta^2 z_i^2$. Obviously the above model can also be written in the form of a product model as

$$y_{ij} = z_i v_{ij}, \tag{2}$$

where v's are independent of z's, and normally distributed with mean 1 and variance δ^2. In this context the z's which are unobserved like the ε's are called latent variables. The parameter δ represents the coefficient of variation. Although the joint distribution of y's is not readily available in a closed form and far from being normal, some elementary estimates of the three basic parameters μ, σ^2, δ can be derived.

2 Background

Let

$$\begin{aligned} \bar{y}_i &= \textstyle\sum_{j=1}^{k_i} y_{ij}/k_i, \\ \bar{y} &= \textstyle\sum_{i=1}^{n} k_i \bar{y}_i / k_\bullet, \\ BSS &= \textstyle\sum_{i=1}^{n} k_i (\bar{y}_i - \bar{y})^2, \\ WSS &= \textstyle\sum_{i=1}^{n} \sum_{j=1}^{k_i} (y_{ij} - \bar{y}_i)^2, \end{aligned} \tag{3}$$

where $k_\bullet = \sum_{i=1}^{n} k_i$.

Define

$$
\begin{aligned}
k_{2\bullet} &= \textstyle\sum_{i=1}^{n} k_i^2, \\
k_{3\bullet} &= \textstyle\sum_{i=1}^{n} k_i^3, \\
\Xi &= (\mu^2+\sigma^2)^2, \\
\Psi &= \sigma^2(\mu^2+\sigma^2).
\end{aligned} \tag{4}
$$

Then the following results on moments of some elementary statistics can be derived.

Proposition 1.

(a) $E(\bar{y}_i) = \mu$, $Var(\bar{y}_i) = \sigma^2 + \delta^2(\sigma^2+\mu^2)k_i$,

(b) $E(BSS) = (n-1)\delta^2(\sigma^2+\mu^2) + \sigma^2(k_\bullet - \dfrac{k_{2\bullet}}{k_\bullet})$,

(c) $E(WSS) = (k_\bullet - n)\delta^2(\sigma^2+\mu^2)$,

(d)
$$
\begin{aligned}
E(BSS^2) = {} & \delta^4\Big\{(n^2-1)\Xi + 12(n-2+\frac{k_{2\bullet}}{k_\bullet^2})\Psi - 6(n-2+\frac{k_{2\bullet}}{k_\bullet^2})\sigma^4\Big\} \\
& +\delta^2\Big\{2(n+1)(k_\bullet - \frac{k_{2\bullet}}{k_\bullet})\Psi + \frac{12}{k_{2\bullet}}(k_\bullet^3 - 2k_\bullet k_{2\bullet} + k_{3\bullet})\sigma^4\Big\} \\
& +\{k_\bullet^2 - \frac{4k_{3\bullet}}{k_\bullet} + \frac{3(k_{2\bullet})^2}{k_\bullet^2}\}\sigma^4,
\end{aligned}
$$

(e) $E(WSS^2) = \delta^4\Big\{(k_\bullet - n)(k_\bullet - n + 2)\Xi + (k_{2\bullet} - n)(4\Psi - 2\sigma^4)\Big\}$,

(f)
$$
\begin{aligned}
E(BSS\cdot WSS) = {} & \delta^4\Big\{(n-1)(k_\bullet - n)\Xi + (k_\bullet - n + 1 - \frac{k_{2\bullet}}{k_\bullet})(4\Psi - 2\sigma^4)\Big\} \\
& +\delta^2\Big\{(k_\bullet^2 - nk_\bullet - k_{2\bullet} + \frac{nk_{2\bullet}}{k_\bullet})\Psi + 2(k_{2\bullet} - k_\bullet + \frac{k_{2\bullet}}{k_\bullet} - \frac{k_{3\bullet}}{k_\bullet})\sigma^4\Big\}.
\end{aligned}
$$

Although naive and moment estimators of the three parameters have been suggested in the literature (Tsang, 1998) and a relative comparison based entirely on simulation has been carried out, so far no exact expressions for bias and variance of these estimators are available. Tsang (1998) also derived the MLEs of the parameters based on the EM algorithm. The naive estimators are given by

$$\begin{aligned}
\hat{\delta}_{\text{naive}} &= \left[\sum_{i=1}^{n} \frac{k_i s_i^2}{k_\bullet \bar{y}_i^2}\right]^{1/2}, \\
\hat{\mu}_{\text{naive}} &= \bar{y}, \\
\hat{\sigma}^2_{\text{naive}} &= \frac{BSS}{k_\bullet},
\end{aligned} \tag{5}$$

where $s_i^2 = \sum_{j=1}^{k_i} (y_{ij} - \bar{y}_i)^2 / k_i$. The moment estimators of μ is the same as in (5), and those of σ^2 and δ^2 are given by

$$\hat{\sigma}^2_{\text{mom}} = \left(k_\bullet - \frac{k_{2\bullet}}{k_\bullet}\right)^{-1} (n-1) \left(\frac{BSS}{n-1} - \frac{WSS}{k_\bullet - n}\right), \tag{6}$$

$$\hat{\delta}_{\text{mom}} = \left(\frac{WSS}{(k_\bullet - n)(\hat{\mu}^2 + \hat{\sigma}^2)}\right)^{1/2}. \tag{7}$$

It may be mentioned that the above estimators are based on parts (a) – (c) of Proposition 1. It should also be noted that $\hat{\sigma}^2_{\text{mom}}$ can take negative values with a positive probability. Following Chow & Shao (1988), one can use the modified truncated version given by

$$\hat{\sigma}^2_{\text{mom/tr}} = \begin{cases} \left(k_\bullet - \dfrac{k_{2\bullet}}{k_\bullet}\right)^{-1} (n-1) \left(\dfrac{BSS}{n} - \dfrac{WSS}{k_\bullet - n}\right) & \text{for } BSS > \dfrac{n+1}{k_\bullet - n} WSS, \\ \left(k_\bullet - \dfrac{k_{2\bullet}}{k_\bullet}\right)^{-1} (n-1) \dfrac{BSS}{n(n+1)} & \text{for } BSS \le \dfrac{n+1}{k_\bullet - n} WSS. \end{cases}$$

There are several objectives of this paper. Following Mathew, Sinha & Sutradhar (1992), in Section 3 we derive a quadratic estimator of σ^2 of the form $aBSS + bWSS$ which has a uniformly smaller mean squared error compared to $\hat{\sigma}^2_{\text{mom}}$. Details of the derivation are shown in the balanced case, i.e. when $k_1 = \ldots = k_n = k$, and an outline is given in the general unbalanced case. In Section 4 we provide another set of estimators based on a heuristic moment method. Although the estimators are not explicit, a numerical evaluation for any given data set can be performed. Furthermore, in Section 4 we provide a Bayesian perspective of the problem and derive appropriate conditional densities in order to obtain Bayes estimators of the parameters. A comparison of the proposed estimators with the existing estimators in terms of bias and mean squared errors is briefly presented in Section 5.

3 Improved estimator of σ^2

Following parts (d) – (f) of Proposition 1, the variance of the moment estimator $\hat{\sigma}^2_{\text{mom}}$ of σ^2 can be shown to be equal to

$$
\begin{aligned}
&Var(\hat{\sigma}^2_{\text{mom}}) = E\{(\hat{\sigma}^2_{\text{mom}} - \sigma^2)^2\} = E(\hat{\sigma}^4_{\text{mom}}) - \sigma^4 \\
&\quad = \frac{(n-1)^2}{\left(k_\bullet - \frac{k_{2\bullet}}{k_\bullet}\right)^2} E\left\{\frac{BSS^2}{(n-1)^2} + \frac{WSS^2}{(k_\bullet - n)^2} - 2\frac{BSS \cdot WSS}{(k_\bullet - n)(n-1)}\right\} - \sigma^4 \\
&\quad = \left(k_\bullet - \frac{k_{2\bullet}}{k_\bullet}\right)^{-2} \left(\delta^4 \left\{\left[\frac{2(n-1)(k_\bullet - 1)}{k_\bullet - n}\right]\Xi + \left[\frac{12k_{2\bullet}}{k_\bullet^2} + 12n - 24\right.\right.\right. \\
&\quad + \frac{4(n-1)}{(k_\bullet - n)^2}\left(-3n(n-1) - 2k_\bullet(k_\bullet + 1 - 2n) + k_{2\bullet}(1+n) - \frac{2nk_{2\bullet}}{k_\bullet}\right)\Bigg]\Psi \\
&\quad - \left[\frac{6k_{2\bullet}}{k_\bullet^2} + 6n - 12 + \frac{2(n-1)}{(k_\bullet - n)^2}\left(-3n(n-1) - 2k_\bullet(k_\bullet + 1 - 2n)\right.\right. \\
&\quad \left.\left.\left. + k_{2\bullet}(n+1) - \frac{2nk_{2\bullet}}{k_\bullet}\right)\right]\sigma^4\right\} + \delta^2\left\{\left[k_\bullet^2 - k_{2\bullet} - nk_\bullet + \frac{nk_{2\bullet}}{k_\bullet}\right]\frac{2\Psi}{k_\bullet - n}\right. \\
&\quad \left. + \left[k_\bullet(3k_\bullet - 2n - 1) - 5k_{2\bullet} - nk_{2\bullet} + \frac{5nk_{2\bullet}}{k_\bullet} + \frac{k_{3\bullet}}{k_\bullet}(n+2) - \frac{k_{2\bullet}}{k_\bullet}\right]\frac{\sigma^4}{k_\bullet - n}\right\} \\
&\quad \left. + \left\{k_\bullet^2 - \frac{4k_{3\bullet}}{k_\bullet} + \frac{3(k_{2\bullet})^2}{k_\bullet^2}\right\}\sigma^4\right) - \sigma^4
\end{aligned}
$$

which we have written in powers of δ and where Ξ and Ψ were defined in (4). For the balanced model we get

$$
\begin{aligned}
Var(\hat{\sigma}^2_{mom}) &= \frac{1}{k^2} E\left\{\frac{BSS^2}{(n-1)^2} + \frac{WSS^2}{(k_\bullet - n)^2} - 2\frac{BSS \cdot WSS}{(n-1)(k_\bullet - n)}\right\} - \sigma^4 \\
&= \frac{\delta^4}{k^2}\left\{\frac{2(kn-1)}{(k-1)n(n-1)}\Xi + \frac{4k}{(k-1)n}(2\Psi - \sigma^4)\right\} \\
&\quad + \frac{\delta^2}{k^2}\left\{\frac{4k}{n-1}\Psi + \frac{8k}{n}\sigma^4\right\} + \left\{\frac{2}{n-1}\sigma^4\right\}. \qquad (8)
\end{aligned}
$$

We propose $\tilde{\sigma}^2 = aBSS + bWSS$ as a rival estimate of σ^2 and derive values of a and b such that MSE($aBSS + bWSS$) is uniformly smaller than $Var(\hat{\sigma}^2_{mom})$, for all μ, σ^2 and δ^2. Using parts (d) – (f) of Proposition 1 we first evaluate MSE($\tilde{\sigma}^2$) and arrange terms in powers of δ. This is given below for the balanced case. It can be seen that only even powers of δ appear in this expression as in $Var(\hat{\sigma}^2_{\text{mom}})$.

Proposition 2. *The* $MSE(\tilde{\sigma}^2)$ *for the balanced case is given by*

$$\begin{aligned}
MSE(\tilde{\sigma}^2) &= MSE(aBSS + bWSS) = E(aBSS + bWSS - \sigma^2)^2 \\
&= \delta^4 \Bigg\{ a^2[(n^2-1)\Xi + 6\frac{(n-1)^2}{n}(2\Psi - \sigma^4)] \\
&\quad + b^2[(k-1)n(kn-n+2)\Xi + 2n(k^2-1)(2\Psi - \sigma^4)] \\
&\quad + 2ab[(k-1)n(n-1)\Xi + 2(k-1)n(n-1)(2\Psi - \sigma^4)] \Bigg\} \\
&\quad + \delta^2 \Bigg\{ [a^2[2k(n^2-1)\Psi + \frac{12k}{n}(n-1)^2\sigma^4] \\
&\quad + 2ab[k(k-1)(n-1)(n\Psi + 2\sigma^4)] - 2(a(n-1) + b(k-1)n)\Psi \Bigg\} \\
&\quad + \Bigg\{ [a^2k^2(n^2-1) - 2ak(n-1) + 1]\sigma^4 \Bigg\}. \qquad (9)
\end{aligned}$$

Now a and b are chosen so that (9) becomes smaller than (8). Taking δ arbitrarily small and comparing, i.e. comparing the terms which are independent of δ a necessary condition on a is given by

$$a^2k^2(n^2-1) - 2ak(n-1) + 1 \le \frac{2}{n-1}. \qquad (10)$$

Choosing a to minimize the quadratic function in the left hand side of (10) yields

$$a_{\text{opt}} = \frac{1}{k(n+1)}. \qquad (11)$$

Taking δ arbitrarily large and comparing the leading terms in the coefficients of δ^4 in (8) and (9), a necessary condition on a and b is given by

$$a + (k-1)b = 0. \qquad (12)$$

Taking

$$b_{\text{opt}} = -\frac{a_{\text{opt}}}{k-1} \qquad (13)$$

we now prove the following main result.

Theorem 1 $MSE(a_{\text{opt}}BSS + b_{\text{opt}}WSS)$ *is uniformly smaller than* $MSE(\hat{\sigma}^2_{\text{mom}})$.

Proof: We prove in the Appendix that (i) the terms of δ^4 in (9) are uniformly smaller than the corresponding ones of δ^4 in (8), and (ii) the terms of δ^2 in (9) are uniformly smaller than those of δ^2 in (8).

The improved quadratic estimator of σ^2 is given by

$$\tilde{\sigma}^2 = \frac{BSS}{k(n+1)} - \frac{WSS}{k(k-1)(n+1)}. \tag{14}$$

A similar but much more involved argument in the unbalanced case reveals that proper values of a and b are given by

Theorem 2

$$a_{\text{unb}} = \frac{k_\bullet - \dfrac{k_{2\bullet}}{k_\bullet}}{k_\bullet^2 - \dfrac{4k_{3\bullet}}{k_\bullet} + \dfrac{3(k_{2\bullet})^2}{k_\bullet^2}}, \tag{15}$$

$$b_{\text{unb}} = -\frac{a_{\text{unb}}}{\bar{k}-1}, \tag{16}$$

where $\bar{k} = n^{-1}\sum_{i=1}^n k_i$.

Obviously, the above expressions reduce to (11) and (13) in the balanced case.

4 Heuristic and Bayesian estimators

In this section we propose some heuristic estimators of the three parameters in the unbalanced case. Noting parts (a) and (c) of Proposition 1, which clearly reveal that $\bar{y}_i$'s have unequal variances in the unbalanced case, we propose the following three equations in the three unknowns μ, σ^2, δ:

$$\mu = \frac{\sum_{i=1}^n k_i\bar{y}_i[k_i\sigma^2 + \delta^2(\mu^2+\sigma^2)]^{-1}}{\sum_{i=1}^n k_i[k_i\sigma^2 + \delta^2(\mu^2+\sigma^2)]^{-1}}, \tag{17}$$

$$\sum_{i=1}^n \frac{k_i(\bar{y}_i-\mu)^2}{[k_i\sigma^2 + \delta^2(\mu^2+\sigma^2)]} = n-1, \tag{18}$$

$$WSS = (k_\bullet - n)\delta^2(\mu^2+\sigma^2). \tag{19}$$

Plugging in (19) into (17) and (18) results in the following two simpler equations in μ and σ^2:

$$\mu = \frac{\sum_{i=1}^n k_i\bar{y}_i\left(k_i\sigma^2 + \dfrac{WSS}{k_\bullet - n}\right)^{-1}}{\sum_{i=1}^n k_i\left(k_i\sigma^2 + \dfrac{WSS}{k_\bullet - n}\right)^{-1}}, \tag{20}$$

$$\sum_{i=1}^{n} \frac{k_i(\bar{y}_i - \mu)^2}{k_i\sigma^2 + \dfrac{WSS}{k_\bullet - n}} = n - 1. \tag{21}$$

It should be mentioned that in the balanced case the above equations are the same as those given by Tsang (1998).

Let $Y = (y_{ij},\ j = 1, \ldots, k_i,\ i = 1, \ldots, n),\ Z = (z_1, \ldots, z_n),\ \theta = (\mu,\ \sigma^2,\ \delta)$. Assuming non-informative priors, the joint density of $(Y,\ Z,\ \theta)$ is given by

$$\begin{aligned} &f(Y, Z, \theta) \\ &\quad \propto (\sigma^2)^{-n/2-1}\delta^{-k_\bullet-1}\left(\prod_{i=1}^{n} z_i^{-k_i}\right)\exp\left\{-\frac{1}{2\delta^2}\sum_{i=1}^{n}\sum_{j=1}^{k_i}\left(\frac{y_{ij}}{z_i} - 1\right)^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(z_i - \mu)^2\right\}. \end{aligned}$$

Since it is difficult to integrate out the latent variables Z to get the joint density of (Y, θ) and hence the posterior density of θ given Y, we consider the posterior of θ conditional on the augmented data (Y, Z). To get the Bayes estimator of θ, i.e. $E(\theta|Y)$, Markov Chain Monte Carlo Methods such as the Gibbs sampler or the data augmentation algorithm can be used.

Given (Y, Z), we have

$$f(\theta|Y, Z) \propto \delta^{-k_\bullet-1}\exp\left\{-\frac{1}{2\delta^2}\sum_{i=1}^{n}\sum_{j=1}^{k_i}\left(\frac{y_{ij}}{z_i} - 1\right)^2\right\}(\sigma^2)^{-n/2-1}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(z_i - \mu)^2\right\}.$$

Therefore given (Y, Z), δ and (μ, σ^2) are conditional independent. The conditional densities are given by

$$\begin{aligned} f(\delta|Y, Z) &\propto \delta^{-k_\bullet-1}\exp\left\{-\frac{1}{2\delta^2}\sum_{i=1}^{n}\sum_{j=1}^{k_i}\left(\frac{y_{ij}}{z_i} - 1\right)^2\right\}, \\ f(\mu, \sigma^2|Y, Z) &\propto (\sigma^2)^{-n/2-1}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(z_i - \mu)^2\right\} \\ &= (\sigma^2)^{-n/2-1}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(z_i - \bar{z})^2\right\}\exp\left\{-\frac{n}{2\sigma^2}(\mu - \bar{z})^2\right\}. \end{aligned}$$

Integrating out μ in the second expression we get

$$f(\sigma^2|Y, Z) \propto (\sigma^2)^{-\frac{1}{2}(n+1)}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(z_i - \bar{z})^2\right\},$$

and hence

$$f(\mu|\sigma^2, Y, Z) \propto \exp\left\{-\frac{n}{2\sigma^2}(\mu - \bar{z})^2\right\}.$$

From the above expressions we can see that given the augmented data (Y, Z), the distribution of θ is simple and it is very easy to generate θ by observing

$$\begin{aligned}
\delta|_{Y,Z} &\cong \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{k_i}\left(\frac{y_{ij}}{z_i} - 1\right)^2 / \chi^2_{k_\bullet}}, \\
\sigma^2|_{Y,Z} &\cong \sum_{i=1}^{n}(z_i - \bar{z})^2/\chi^2_{n-5}, \\
f(\mu|\sigma^2, Y, Z) &\propto \exp\left\{-\frac{n}{2\sigma^2}(\mu - \bar{z})^2\right\}.
\end{aligned}$$

Now consider the conditional density of Z given (Y, θ),

$$\begin{aligned}
f(Z|Y,\theta) &\propto \left(\prod_{i=1}^{n} z_i^{-k_i}\right)\exp\left\{-\frac{1}{2\delta^2}\sum_{i=1}^{n}\sum_{j=1}^{k_i}\left(\frac{y_{ij}}{z_i} - 1\right)^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(z_i - \mu)^2\right\} \\
&= \prod_{i=1}^{n}\left\{z_i^{-k_i}\exp\left\{-\frac{1}{2\delta^2}\sum_{j=1}^{k_i}\left(\frac{y_{ij}}{z_i} - 1\right)^2 - \frac{1}{2\sigma^2}(z_i - \mu)^2\right\}\right\}.
\end{aligned}$$

Therefore z_i, $i = 1, \ldots, n$, are conditionally independent given (Y, θ). The individual densities are given by

$$f(z_i|Y,\theta) \propto z_i^{-k_i}\exp\left\{-\frac{1}{2\delta^2}\sum_{j=1}^{k_i}\left(\frac{y_{ij}}{z_i} - 1\right)^2 - \frac{1}{2\sigma^2}(z_i - \mu)^2\right\}.$$

Using a suitable acceptance rejection algorithm we can generate z_i via $f(Z|Y,\theta)$. Alternatively, since z_i is one-dimensional we can by applying numerical integration draw from the distribution of z_i. Therefore we can draw from $f(\theta|Y, Z)$ and $f(Z|Y,\theta)$ easily.

5 Comparisons of some of the proposed estimators

Before making any comparison between the new estimator $\tilde{\sigma}^2$ and $\hat{\sigma}^2_{\text{mom}}$ bias has to be investigated.

Theorem 3 *For the improved quadratic estimator given in Theorem 1*

$$\begin{aligned}
Bias &= E(\tilde{\sigma}^2) - \sigma^2 = E\left(\frac{BSS}{k(n+1)} - \frac{WSS}{k(k-1)(n+1)}\right) - \sigma^2 \\
&= -\frac{\delta^2\mu^2}{k(n+1)} + \sigma^2\left(-\frac{\delta^2}{k(n+1)} + \frac{n-1}{n+1}\right) - \sigma^2 \\
&= -\frac{\delta^2\mu^2}{k(n+1)} - \sigma^2\left(\frac{\delta^2}{k(n+1)} + \frac{2}{n+1}\right). \qquad (22)
\end{aligned}$$

From Theorem 3 it follows that $Bias < 0$ for all μ and δ, i.e. $\tilde{\sigma}^2$ underestimates the parameter σ^2. If $n \to \infty$ then $Bias \to 0$. For large k it follows from (23) that $Bias$ is solely a function of σ^2. It is interesting to compare $\tilde{\sigma}^2$ and σ^2_{mom}.

Theorem 4 *In the balanced case it follows that*

$$\begin{aligned}
\hat{\sigma}^2_{\text{mom}} &= \frac{BSS}{k(n-1)} - \frac{WSS}{k(k-1)n} = \frac{1}{k(n-1)}\left(BSS - \frac{WSS}{k-1} + \frac{WSS}{n(k-1)}\right), \\
\tilde{\sigma}^2 &= \frac{BSS}{k(n+1)} - \frac{WSS}{k(k-1)(n+1)} = \frac{1}{k(n+1)}\left(BSS - \frac{WSS}{k-1}\right)
\end{aligned}$$

and thus $\tilde{\sigma}^2 < \hat{\sigma}^2_{\text{mom}}$.

If bias is strong then $\tilde{\sigma}^2$ is meaningless to use. Furthermore, one may question the use of the model if σ^2 is small in comparison to δ^2, i.e. Z_i does not vary much. We have performed a number of simulations and $\tilde{\sigma}^2$ performs better than both $\hat{\sigma}^2_{\text{mom}}$ and its truncated version $\hat{\sigma}^2_{\text{mom/tr}}$. However, this result is only valid if both $\hat{\sigma}^2_{\text{mom}}$ and $\tilde{\sigma}^2$ are positive and bias is not severe. Unfortunately we were not able to explicitly calculate the bias of $\hat{\sigma}^2_{\text{mom/tr}}$.

Turning to a comparison with the Bayesian estimator we first note that there is a different basis on which the Bayesian paradigm relies and thus comparisons are not fully interpretable. In any case, in our simulations, if data indicates a clear variation in Z_i the Bayesian estimator based on MCMC methodology performs somewhat better than the others. On the other hand the computations are cumbersome, even with only three parameters. The other estimators are working better when the variation in Z_i becomes small or δ^2 becomes larger. With a more sophisticated Bayesian program some of the drawbacks may be reduced.

Appendix

Here is given a proof of Theorem 1, i.e. (8) and (9) are compared. It will be shown that (9) is uniformly smaller than (8) with respect to the parameters Ξ, Ψ and σ^4. We start by studying the terms of δ^4. First the terms for Ξ are compared, i.e. we are going to show that

$$\frac{\frac{2kn}{k-1}-1}{(n+1)^2} \leq \frac{2(nk-1)}{n(k-1)(n-1)}$$

which is equivalent to

$$2nk \leq k-1+\frac{2(n+1)^2(nk-1)}{n(n-1)}.$$

The right-hand side equals

$$\begin{aligned} k-1+2(nk-1)+\frac{4(nk-1)}{n-1}&+\frac{2(n+1)(nk-1)}{n(n-1)} \\ &\geq 2nk+k-1-2+4 \geq 2nk. \end{aligned}$$

Now we study the terms of $(2\Psi-\sigma^4)$:

$$\frac{1}{(n+1)^2}\left[\frac{2}{n}(n-1)(n-3)+\frac{2(k+1)n}{k-1}\right] \leq \frac{4k}{n(k-1)}$$

which is equivalent to

$$(n^2-4n+3)(k-1)+n^2(k+1) \leq 2k(n+1)^2$$

and by expanding the brackets it is immediately seen that this is true.

It will be verified that the terms of δ^2 in $MSE(\tilde{\sigma}^2)$ are less or equal to the coefficient of δ^2 in $MSE(\hat{\sigma}^2_{mom})$ for all Ψ and σ^4, i.e. the following inequality holds:

$$\begin{aligned} a^2\{2k(n^2-1)\Psi+\frac{12k}{n}(n-1)^2\sigma^4\}+2ab\{k(k-1)(n-1)(n\Psi+2\sigma^4)\}& - \\ -2\{a(n-1)+bn(k-1)\}\Psi \leq \frac{4k}{n-1}\Psi+\frac{8k}{n}\sigma^4&, \end{aligned}$$

where a and b are given by (11) and (13), respectively. For the terms connected to Ψ we have to show that

$$\frac{2(n^2-1)}{(n+1)^2}-\frac{2(n-1)n}{(n+1)^2}-\frac{2(n-1)}{n+1}+\frac{2n}{n+1} \leq \frac{4}{n-1}$$

which is equivalent to

$$\frac{4n}{(n+1)^2} \leq \frac{4}{n-2}$$

which is true for all n. For the coefficient σ^4 it can easily be shown that for all n

$$\frac{12(n-1)^2}{n(n+1)^2} - \frac{4(n-1)}{(n+1)^2} \leq \frac{8}{n}$$

holds.

Finally it follows from (8) and (9) that we have to show that

$$\frac{n^2-1}{(n+1)^2} - \frac{2(n-1)}{n+1} + 1 \leq \frac{2}{n-1}$$

which by some manipulations is shown to be equivalent to

$$1 \leq \frac{n+1}{n-1}.$$

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