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On a Class of Singular Nonsymmetric Matrices with Nonnegative Integer Spectra

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Abstract

The objective of this paper is to consider a class of singular nonsymmetric matrices with integer spectrum. The class comprises generalized triangular matrices with diagonal elements obtained by summing the elements of the corresponding column. If the size of a matrix belonging to the class equals $n \times n$, the spectrum of the matrix is given by the sequence of distinct non-negative integers up to $n-1$, irrespective of the elements of the matrix. Right and left eigenvectors are obtained. Moreover, several interesting relations are presented, including factorizations via triangular matrices.

Keywords: Eigenvectors, generalized triangular matrix, integer spectrum, nonsymmetric matrix, triangular factorization.

AMS classification: 15A18, 15A23.

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1 Introduction

In this paper we consider a new class of singular matrices with remarkable algebraic properties. For example, the spectrum of a matrix belonging to this class depends only on the size of the matrix and not on the specific elements of this matrix. Moreover, the spectrum entirely consists of successive non-negative integer values $0, 1, \ldots, n-1$. A special case of this class of matrices originates from statistical sampling theory (Bondesson & Traat, 2005, 2007).

In their papers, via sampling theory (the Poisson sampling design) as well as some analytic proofs, eigenvalues and eigenvectors of these matrices were presented. Their proofs remind on the use of Lagrangian polynomials which for example are used when finding the inverse of a Vandermonde matrix (see e.g. Macon & Spitzbart, 1958; El-Mikkawy, 2003). We have not found any other work related to the matrix class which we are going to consider. The main purpose of this paper is to introduce the class, show some basic algebraic properties, show how to factor the class and demonstrate how to find eigenvalues and eigenvectors of matrices belonging to the class. For the derivation of the right eigenvectors a special Vandermonde matrix plays an important role.

Consider a strictly upper triangular matrix of ones, i.e

$$\begin{pmatrix} 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & 1 & \ldots & 1 & 1 \\ 0 & 0 & 0 & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & 1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

and put the sum of each column on the main diagonal. This yields matrices of the form

$$\begin{pmatrix} 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & 2 & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & 1 & 1 \\ 0 & 0 & 0 & \ldots & n-2 & 1 \\ 0 & 0 & 0 & \ldots & 0 & n-1 \end{pmatrix}. \tag{1.1}$$

One straightforward result is that the eigenvalues of the matrix in (1.1) are given by $0, 1, 2, \ldots, n-1$.

We are going to study the following generalized version of the matrix presented in (1.1).

Definition 1.1 *A square nonsymmetric matrix $B = (b_{ij})$ of order n belongs to the $\mathbb{B}_n$-class if its elements satisfy the following conditions:*

$$b_{ii} = \sum_{j=1, j\neq i}^{n} b_{ji}, \quad i = 1, \dots, n, \tag{1.2}$$

$$b_{ij} + b_{ji} = 1, \quad j \neq i,\ i, j = 1, \dots, n, \tag{1.3}$$

$$b_{ij} - b_{ik} = \frac{b_{ij} b_{ki}}{b_{kj}},\ b_{kj} \neq 0, \quad j \neq k,\ i \neq k, j;\ i, j, k = 1, \dots, n. \tag{1.4}$$

Instead of (1.4) one may use $b_{kj} = \frac{b_{ij} b_{kj}}{b_{ij} - b_{ik}}$ or $b_{ij} b_{kj} = b_{ik} b_{kj} + b_{ij} b_{ki}$. Relation (1.3) defines a generalized triangular structure and it can be shown that (1.4) is a necessary and sufficient condition for the class to have the non-negative integer spectra consisting of the distinct integers $\{0, 1, \dots, n-1\}$. Due to (1.2), the sum of the elements in each row equals $n - 1$.

It may be noted that another matrix with integer eigenvalues and row element sum equal to $n - 1$, with many applications in various fields, is the well known tridiagonal Kac-matrix (Clement matrix); see Taussky & Todd (1991). Moreover, for any $B \in \mathbb{B}_n$ with positive elements we may consider $(n-1)^{-1} B$ as a transition matrix with interesting symmetric properties reflected by the equidistant integer spectra.

When $B \in \mathbb{B}_3$,

$$\begin{aligned} B &= \begin{pmatrix} b_{21} + b_{31} & b_{12} & b_{13} \\ b_{21} & b_{12} + b_{32} & b_{23} \\ b_{31} & b_{32} & b_{13} + b_{23} \end{pmatrix} \\ &= \begin{pmatrix} b_{21} + b_{31} & 1 - b_{21} & 1 - b_{31} \\ b_{21} & 1 - b_{21} + b_{32} & 1 - b_{32} \\ b_{31} & b_{32} & 2 - b_{31} - b_{32} \end{pmatrix}. \end{aligned} \tag{1.5}$$

A general $B \in \mathbb{B}_n$ can be written as

$$B = \begin{pmatrix} \sum_{i=2}^{n} b_{i1} & b_{12} & \dots & b_{1n} \\ b_{21} & \sum_{\substack{i=1,\\ i\neq 2}}^{n} b_{i2} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & \sum_{i=1}^{n-1} b_{in} \end{pmatrix},$$

with elements satisfying (1.3)–(1.4).

It is worth observing that any $B \in \mathbb{B}_n$ is a sum of three matrices: an upper triangular matrix, a diagonal matrix and a skew-symmetric matrix. For (1.5) we have

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} b_{21}+b_{31} & 0 & 0 \\ 0 & -b_{21}+b_{32} & 0 \\ 0 & 0 & -b_{31}-b_{32} \end{pmatrix} + \begin{pmatrix} 0 & -b_{21} & -b_{31} \\ b_{21} & 0 & -b_{32} \\ b_{31} & b_{32} & 0 \end{pmatrix}.$$

Note that the eigenvalues $\{0, 1, 2\}$ of B are found on the diagonal of the upper triangular matrix, irrespective of the values of (b_{ij}) as long as they satisfy (1.2)–(1.4).

In the Conditional Poisson sampling design (e.g., see Aires, 1999),

$$b_{ij} = \frac{p_i(1-p_j)}{p_i - p_j}$$

are used to calculate conditional inclusion probabilities, where the p_i's are inclusion probabilities under the Poisson design. Bondesson & Traat (2005, 2007) generalized this expression somewhat and considered

$$b_{ij} = \frac{r_i}{r_i - r_j}, \tag{1.6}$$

where the r_i's are arbitrary distinct values. In this paper, instead of (1.6), we assume (1.4) to hold. Note that any b_{ij} satisfying (1.6) also satisfies (1.4). For the matrix defined via the elements in (1.6) Bondesson & Traat (2005, 2007) presented eigenvalues, and right and left eigenvectors. They expressed their results as functions of r_i in (1.6) whereas in this paper we will express the results in terms of b_{ij}, i.e. the elements of $B \in \mathbb{B}_n$. Moreover, the proof of all results in this paper are pure algebraic whereas Bondesson & Traat (2005, 2007) indicated proofs based on series expansions and identification of coefficients. It is however not clear how to apply their results to the $\mathbb{B}_n$-class of matrices, given in Definition 1.1. Moreover, the algebraic approach of this paper opens up a world of interesting relations. In particular, the triangular factorization of matrices in the $\mathbb{B}_n$-class, presented in Section 4.

As noted before it follows from (1.4) that

$$b_{kj} = \frac{b_{ij}b_{ki}}{b_{ij} - b_{ik}} = \frac{b_{ij}(1-b_{ik})}{b_{ij} - b_{ik}}. \tag{1.7}$$

Hence, any row in B, $B \in \mathbb{B}_n$, generates all other elements and thus, there are at most $n-1$ functionally independent elements in B. For example, we may use b_{1j},

$j = 2, 3, \ldots, n$, to generate all other elements in B. Furthermore, if we choose for r_j in (1.6), without loss of generality, $r_1 = 1$ and

$$r_j = -\frac{b_{j1}}{b_{ij}}, \quad j = 2, 3, \ldots, n,$$

it follows that

$$b_{1j} = \frac{1}{1 - r_j}$$

and

$$b_{ij} = \frac{\frac{1}{1-r_j}(1 - \frac{1}{1-r_i})}{\frac{1}{1-r_j} - \frac{1}{1-r_i}} = \frac{r_i}{r_i - r_j}.$$

Thus, all b_{ij}'s can be generated by the above choice of r_j. This means that a matrix defined by (1.6), as considered in Bondesson & Traat (2005, 2007), is a canonical version of any matrix defined through (1.4), assuming that (1.2) and (1.3) hold.

In the next we generalize the class $\mathbb{B}_n$.

Definition 1.2 *The matrix $B_{n,k}$: $(n-k+2) \times (n-k+2)$, $k = 2, \ldots, n$, is obtained from the matrix B, $B \in \mathbb{B}_n$, by elimination of $(k-2)$ consecutive rows and columns starting from the second row and column, with corresponding adjustments in the main diagonal.*

Note that $B_{n,2} = B$. Two examples of $B_{n,k}$, given in Definition 1.2, are

$$B_{5,3} = \begin{pmatrix} \sum_{i=3}^{5} b_{i1} & b_{13} & b_{14} & b_{15} \\ b_{31} & \sum_{\substack{i=1,\\ i\neq 2,3}}^{5} b_{i3} & b_{34} & b_{35} \\ b_{41} & b_{43} & \sum_{\substack{i=1,\\ i\neq 2,4}}^{5} b_{i4} & b_{45} \\ b_{51} & b_{53} & b_{54} & \sum_{\substack{i=1,\\ i\neq 2}}^{4} b_{i5} \end{pmatrix}, \quad B_{5,4} = \begin{pmatrix} b_{41} + b_{51} & b_{14} & b_{15} \\ b_{41} & b_{14} + b_{52} & b_{45} \\ b_{51} & b_{52} & b_{15} + b_{45} \end{pmatrix}.$$

Moreover, in this paper we consider only real-valued matrices, although the generalization to matrices with complex-valued entries could be performed fairly easy.

The paper consists of five sections. In Section 2 some basic and fundamental relations for any $B \in \mathbb{B}_n$, which will be used in the subsequent, are given. Section 3 consists of a straightforward proof concerning the spectrum of any $B \in \mathbb{B}_n$. In Section 4 we consider a factorization of $B \in \mathbb{B}_n$ into a product of three triangular matrices. Finally, in Section 5 expressions of left and right eigenvectors are presented. The most general expressions hold for any size of the matrix B. At some few places where calculations become too lengthy only brief descriptions of the proofs are indicated. Instead, in these cases a number of detailed calculations are shown for matrices of smaller size.

2 Preparations

In this section some relations between the elements in $B \in \mathbb{B}_n$ are shown to hold which all are of utmost importance for the subsequent presentation.

Theorem 2.1 *Let $B \in \mathbb{B}_n$. For all $n > 1$*

(i) *The sum of the products of the off-diagonal row elements equals 1:*

$$\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} b_{ij} = 1.$$

(ii) *The sum of the products of the off-diagonal column elements equals 1:*

$$\sum_{j=1}^{n} \prod_{\substack{i=1 \\ i \neq j}}^{n} b_{ij} = 1.$$

Proof: Because of symmetry only (i) is proven. For $n = 2$ we obtain the trivial relation $b_{12} + b_{21} = 1$. Moreover, for $n = 3$

$$\begin{aligned} \sum_{i=1}^{3} \prod_{\substack{j=1 \\ j \neq i}}^{3} b_{ij} = b_{12}b_{13} + b_{21}b_{23} + b_{31}b_{32} &= b_{12} - b_{12}b_{31} + b_{21} - b_{21}b_{32} + b_{31}b_{32} \\ &= 1 - (b_{12} - b_{13})b_{32} - b_{21}b_{32} + b_{31}b_{32} \\ &= 1 - (b_{12} + b_{21})b_{32} + (b_{13} + b_{31})b_{32} = 1, \end{aligned}$$

where in the second equality (1.4) is utilized. Now it is assumed that the theorem is true for $n-1$, i.e.

$$\sum_{i=1}^{n-1} \prod_{\substack{j=1 \\ j\neq i}}^{n-1} b_{ij} = 1, \tag{2.1}$$

which by symmetry yields

$$\sum_{\substack{i=1 \\ i\neq k}}^{n} \prod_{\substack{j=1 \\ j\neq i \\ j\neq k}}^{n} b_{ij} = 1, \qquad k = 1, 2, \ldots, n. \tag{2.2}$$

From here on a chain of calculations is started:

$$\begin{aligned}
\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} b_{ij} &= \sum_{i=1}^{n-1} \prod_{\substack{j=1 \\ j\neq i}}^{n-1} b_{ij} b_{in} + \prod_{j=1}^{n-1} b_{nj} \\
&= \sum_{i=1}^{n-2} \prod_{\substack{j=1 \\ j\neq i}}^{n-2} b_{ij} b_{in-1} b_{in} + \prod_{j=1}^{n-2} b_{n-1j} b_{n-1n} + \prod_{j=1}^{n-2} b_{nj} b_{nn-1} \\
&= \sum_{i=1}^{n-2} \prod_{\substack{j=1 \\ j\neq i}}^{n-2} b_{ij} b_{in-1}(1 - b_{ni}) + \prod_{j=1}^{n-2} b_{n-1j}(1 - b_{nn-1}) + \prod_{j=1}^{n-2} b_{nj} b_{nn-1}. \qquad (2.3)
\end{aligned}$$

Since by the induction assumption

$$\sum_{i=2}^{n-2} \prod_{\substack{j=1 \\ j\neq i}}^{n-2} b_{ij} b_{in-1} + \prod_{j=1}^{n-2} b_{n-1j} = 1$$

the last expression in (2.3) equals

$$1 - \sum_{i=1}^{n-2} \prod_{\substack{j=1 \\ j\neq i}}^{n-2} b_{ij}(b_{in-1} - b_{in}) b_{nn-1} - \prod_{j=1}^{n-2} b_{n-1j} b_{nn-1} + \prod_{j=1}^{n-2} b_{nj} b_{nn-1}, \tag{2.4}$$

where (1.4) has been used: $b_{in-1}b_{ni} = (b_{in-1} - b_{in})b_{nn-1}$. Reshaping (2.4) we obtain

$$1 - \sum_{i=1}^{n-1} \prod_{\substack{j=1 \\ j\neq i}}^{n-1} b_{ij} b_{nn-1} + \sum_{\substack{i=1 \\ i\neq n-1}}^{n} \prod_{\substack{j=1 \\ j\neq i \\ j\neq n-1}}^{n} b_{ij} b_{nn-1}, \tag{2.5}$$

and using the induction assumption, i.e. (2.1) as well as (2.2), we see that (2.5) is indeed equal to

$$1 - b_{nn-1} + b_{nn-1} = 1,$$

and the theorem is proved. □

Corollary 2.1 *Let $B \in \mathbb{B}_n$. For all $n > 1$,*

$$\sum_{i=1}^{n-1} \prod_{\substack{j=1 \\ j \neq i}}^{n} b_{ij} = 1 - \prod_{j=1}^{n-1} b_{nj}.$$

Corollary 2.2 *Let $B \in \mathbb{B}_n$. For every integer a such that $a < n$,*

$$\sum_{i=a}^{n} \prod_{\substack{j=a \\ j \neq i}}^{n} b_{ij} = 1.$$

The next theorem presents some basic and easily proven relations which will be applied in several proofs of subsequent theorems.

Theorem 2.2 *Let $B \in \mathbb{B}_n$ and put $c_{ij} = b_{ij}^{-1} b_{ji}$. Then,*

(i) $c_{ij}^{-1} = c_{ji}, \quad i \neq j,$

(ii) $c_{ki} c_{jk} = -c_{ji}, \quad k \neq i,\ j \neq k,\ i \neq j,$ *(cancellation)*

(iii) $c_{ki} c_{lj} = c_{kj} c_{li}, \quad k \neq i, j;\ l \neq i, j.$ *(exchangeability)*

Proof: (i) follows immediately from the definition of c_{ij}. For (ii) we observe that (see (1.4))

$$\frac{b_{ij} b_{ki}}{b_{kj}} = -\frac{b_{ji} b_{ik}}{b_{jk}}$$

and hence,

$$c_{ki} c_{jk} = b_{ki}^{-1} b_{ik} b_{jk}^{-1} b_{kj} = b_{ki}^{-1} b_{ik} b_{jk}^{-1} b_{kj} b_{ij} b_{ij}^{-1} = -b_{ki}^{-1} b_{ki} b_{jk}^{-1} b_{jk} b_{ij} b_{ij}^{-1} = -c_{ji}.$$

Concerning (iii) we note that

$$c_{ki} c_{lj} = c_{ki} c_{lj} c_{il} c_{li} = -c_{ki} c_{ij} c_{li} = c_{kj} c_{li}.$$

□

Throughout the paper we will use the following abbreviations for two types of multiple sums which both frequently will be applied in the subsequent:

$$\sum_{i_1 \le \cdots \le i_k}^{[m,n]} = \sum_{i_1=m}^{n} \sum_{i_2=i_1}^{n} \cdots \sum_{i_k=i_{k-1}}^{n}, \qquad i_0 = m,$$

$$\sum_{i_1 < \cdots < i_k}^{[m,n]} = \sum_{i_1=m}^{n-k+1} \sum_{i_2=i_1+1}^{n-k+2} \cdots \sum_{i_k=i_{k-1}+1}^{n}, \qquad i_0 = m-1.$$

This section is concluded by an ancillary result which because of lengthy calculations will not be proven in detail, besides two special cases.

Theorem 2.3 *Let $B \in \mathbb{B}_n$. Then,*

$$b_{i+11} + \sum_{k=i+1}^{n-1} \sum_{i_1 \le \cdots \le i_{k-1}}^{[k+1,n]} \prod_{m=i+1}^{k} b_{i_{m-i}m}^{-1} b_{mi_{m-i}} b_{1m} b_{k+11} = \prod_{m=i+1}^{n} b_{m1}, \tag{2.6}$$

$$i = 1, 2, \ldots, n-2.$$

Proof: We start the proof by studying the proposed theorem when $i = n-2$ or $i = n-3$. For $i = n-2$, i.e. $n = i+2$, we have that $j = n-1$ and

$$b_{n-11} + b_{nn-1}^{-1} b_{n-1n} b_{1n-1} b_{n1} = b_{n-11} + b_{nn-1}^{-1} b_{n-1n} b_{1n-1} b_{1n}^{-1} b_{1n} b_{n1}$$
$$= b_{n-11} - b_{n-1n} b_{1n} = b_{n-1n} b_{n1}.$$

For $i = n-3$, i.e. $n = i+3$,

$$b_{n-21} + \sum_{j=n-2}^{n-1} \sum_{i_1 \le \cdots \le i_{j-n+3}}^{[j+1,n]} \prod_{m=n-2}^{j} b_{i_{m-i}m}^{-1} b_{mi_{m-i}} b_{1m} b_{j+11}$$
$$= b_{n-21} + b_{nn-2}^{-1} b_{n-2n} b_{1n-2} b_{nn-1}^{-1} b_{n-1n} b_{1n-1} b_{n1}$$
$$+ b_{n-1n-2}^{-1} b_{n-2n-1} b_{1n-2} b_{n-11} + b_{nn-2}^{-1} b_{n-2n} b_{1n-2} b_{n-11}$$
$$= b_{n-21} - b_{nn-2}^{-1} b_{n-2n} b_{1n-2} b_{n-11} b_{1n} - b_{n-21} b_{1n-1} + b_{nn-2}^{-1} b_{n-2n} b_{1n-2} b_{n-11}$$
$$= b_{n-21} b_{n-11} + b_{nn-2}^{-1} b_{n-2n} b_{1n-2} b_{n-11} b_{n1}$$
$$= b_{n-21} b_{n-11} - b_{n-21} b_{n-11} b_{1n} = b_{n-21} b_{n-11} b_{n1}.$$

Suppose that (2.6) holds for $n-1$, and a series of calculations will be started from the left hand side of (2.6) which can be written

$$
\begin{aligned}
& b_{2i+1} + \sum_{k=i+1}^{n-2} \sum_{i_1 \le \cdots \le i_{k-1}}^{[k+1,n-1]} \prod_{m=i+1}^{k} b^{-1}_{i_{m-1}m} b_{mi_{m-1}} b_{1m} b_{k+11} \\
& + \prod_{m=i+1}^{n-1} b^{-1}_{nm} b_{mn} b_{1m} b_{n1} + \sum_{k=i+1}^{n-2} \prod_{m=i+1}^{k} b^{-1}_{nm} b_{mn} b_{1m} b_{n1} \\
& + \sum_{k=i+1}^{n-2} \sum_{i_1=k+1}^{n-1} b^{-1}_{i_1 2} b_{2i_1} b_{12} b_{k+11} \prod_{m=i+2}^{k} b^{-1}_{nm} b_{mn} b_{1m} \\
& + \ldots + \sum_{k=i+1}^{n-2} \sum_{i_1 \le \cdots \le i_{k-1}}^{[k+1,n-1]} \prod_{m=2}^{k-2} b^{-1}_{i_{m-1}m} b_{mi_{m-1}} b_{1m} b_{k+11} b^{-1}_{nk} b_{kn} b_{1k}. \qquad (2.7)
\end{aligned}
$$

Moreover,

$$
\prod_{m=i+1}^{n-1} b^{-1}_{nm} b_{mn} b_{nm} b_{n1} = \prod_{m=i+1}^{n} b_{m1} - \sum_{k=i+1}^{n-1} \prod_{m=i+1}^{k-1} b^{-1}_{nm} b_{mn} b_{1m} \prod_{\underline{m}=k}^{n-1} b_{\underline{m}1}, \qquad (2.8)
$$

which is inserted in (2.6).

Now we sum up. By assumption

$$
b_{2i+1} + \sum_{i_1 \le \cdots \le i_{k-1}}^{[k+1,n-1]} \prod_{m=i+1}^{k} b^{-1}_{i_{m-1}m} b_{mi_{m-1}} b_{1m} b_{k+11} = \prod_{m=i+1}^{n-1} b_{m1}, \qquad (2.9)
$$

which vanishes in (2.7) when taking into account the expression in (2.8) which corresponds to $k = i+1$, and by rather lengthy calculations, except $\prod_{m=i+1}^{n} b_{m1}$ in (2.8) one may show that all other expression sum to zero. This establishes (2.6). □

3 Preliminaries

The aim of this section is to derive, in a very elementary algebraic way, the eigenvalues of the matrix $B \in \mathbb{B}_n$. Moreover, the section serves as an introduction to Section 4, i.e. a similar technique for proving the next theorem will also be used in Section 4.

Throughout the paper we denote by e_i, $i = 1, \ldots, n$, a standard unit vector of size n, i.e. the ith column of the identity matrix I_n. Define the following lower triangular matrix

$$U = e_1e_1' + b_{12}(e_2e_1' - e_2e_2') + \sum_{i=3}^{n} b_{i2}(e_ie_i' - e_ie_2') \tag{3.1}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ b_{12} & -b_{12} & 0 & 0 & 0 & \dots & 0 \\ 0 & -b_{32} & b_{32} & 0 & 0 & \dots & 0 \\ 0 & -b_{42} & 0 & b_{42} & 0 & \dots & 0 \\ 0 & -b_{52} & 0 & 0 & b_{52} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & -b_{n2} & 0 & 0 & 0 & \dots & b_{n2} \end{pmatrix}$$

with inverse

$$U^{-1} = \sum_{i=1}^{n} e_ie_1' - b_{12}^{-1}\sum_{i=2}^{n} e_ie_2' + \sum_{i=3}^{n} b_{i2}^{-1}e_ie_i' \tag{3.2}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & -b_{12}^{-1} & 0 & 0 & 0 & \dots & 0 \\ 1 & -b_{12}^{-1} & b_{32}^{-1} & 0 & 0 & \dots & 0 \\ 1 & -b_{12}^{-1} & 0 & b_{42}^{-1} & 0 & \dots & 0 \\ 1 & -b_{12}^{-1} & 0 & 0 & b_{52}^{-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 1 & -b_{12}^{-1} & 0 & 0 & 0 & \dots & b_{n2}^{-1} \end{pmatrix}.$$

The expression in (3.2) is true because

$$U\{\sum_{i=1}^{n} e_ie_1' - b_{12}^{-1}\sum_{i=2}^{n} e_ie_2' + \sum_{i=3}^{n} b_{i2}^{-1}e_ie_i'\} = e_1e_1' + e_2e_2' + \sum_{i=3}^{n} e_ie_i' = I_n.$$

Theorem 3.1 *Let $B = B_{n,2} \in \mathbb{B}_n$. Then there exists a nonsingular lower triangular matrix U, given in* (3.1)*, such that $UB_{n,2}U^{-1}$ has the following structure*

$$B_{n,2} = \left(\begin{array}{c|c} n-1 & -\sum_{j=1}^{n} b_{1j}b_{12}^{-1} \quad \sum_{k=3}^{n} b_{1k}b_{k2}^{-1}d_k' \\ \hline 0 & B_{n,3} \end{array}\right),$$

where d_k is a standard unit vector of size $n-2$ and $B_{n,3}$: $(n-1)\times(n-1)$ is defined in Definition 1.2.

Proof: First notice that $B_{n,2}$ can be written

$$B_{n,2} = \sum_{j=1}^{n}\sum_{\substack{i=1 \\ i\neq j}}^{n} b_{ji}e_ie_i' + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n} b_{ij}e_ie_j'.$$

Thus

$$
\begin{aligned}
B_{n,2}U^{-1} \quad = \quad & \sum_{j=1}^{n}\sum_{\substack{i=1\\ i\neq j}}^{n} b_{ji}e_ie_1' + \sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n} b_{ij}e_ie_1' - \sum_{j=1}^{n}\sum_{\substack{k=2\\ k\neq j}}^{n} b_{jk}b_{12}^{-1}e_ke_2' \\
(3.3) \qquad & -\sum_{k=1}^{n}\sum_{\substack{j=2\\ j\neq k}}^{n} b_{kj}b_{12}^{-1}e_ke_2' + \sum_{j=1}^{n}\sum_{\substack{k=3\\ k\neq j}}^{n} b_{jk}b_{k2}^{-1}e_ke_k' + \sum_{j=1}^{n}\sum_{k=3}^{n} b_{jk}b_{k2}^{-1}e_je_k'.
\end{aligned}
$$

Premultiplying (3.3) by e_1e_1' yields

$$
(3.4) \qquad (n-1)e_1e_1' - \sum_{j=2}^{n} b_{1j}b_{12}^{-1}e_1e_2' + \sum_{k=3}^{n} b_{1k}b_{k2}^{-1}e_1e_k'
$$

and premultiplying (3.3) by $b_{12}(e_2e_1' - e_2e_2')$ results in

$$
\begin{aligned}
&\sum_{\substack{j=1\\ j\neq 2}}^{n} b_{j2}e_2e_2' - \sum_{j=2}^{n} b_{1j}e_2e_2' + \sum_{j=3}^{n} b_{2j}e_2e_2' + \sum_{k=3}^{n} b_{1k}b_{k2}^{-1}b_{12}e_2e_k' - \sum_{k=3}^{n} b_{2k}b_{k2}^{-1}b_{12}e_2e_k' \\
&\qquad = \sum_{j=3}^{n} b_{j1}e_2e_2' + \sum_{k=3}^{n} b_{1k}e_2e_k',
\end{aligned}
$$

where in the last equality properties (1.3)−(1.4) have been used. It remains to premultiply (3.3) with $\sum_{i=3}^{n} b_{i2}(e_ie_i' - e_ie_2')$ which gives

$$
\sum_{i=3}^{n} b_{i1}e_ie_2' + \sum_{i=3}^{n}\sum_{\substack{j=3\\ j\neq i}}^{n} b_{ji}e_ie_i' + \sum_{i=3}^{n}\sum_{\substack{k=3\\ k\neq i}}^{n} b_{ik}e_ie_k'.
$$

Hence, by summarizing the above calculations we obtain

$$
UB_{n,2}U^{-1} = \left(\begin{array}{c|cc} n-1 & -\sum_{j=1}^{n} b_{1j}b_{12}^{-1} & \sum_{k=3}^{n} b_{1k}b_{k2}^{-1}d_k' \\ \hline 0 & & B_{n,3} \end{array}\right)
$$

which is the statement of the theorem. □

Thus, a useful recursive relation between $B_{n,2}$ and $B_{n,3}$ has been established. An important consequence of Theorem 3.1 is the following result which indeed was our starting point of interest for studying $B \in \mathbb{B}_n$.

Theorem 3.2 *Let $B \in \mathbb{B}_n$. Then B has the nonnegative integer eigenvalues $\{0, 1, \ldots, n-1\}$.*

Let us now introduce the following auxiliary matrices: a lower unit triangular matrix L_n: $n \times n$, with all non-zero elements equal to one, and a lower bidiagonal matrix K_n with 1 in the main diagonal and -1 in the the first subdiagonal which actually is the inverse of L_n, i.e. $K_n = L_n^{-1}$.

Theorem 3.3 *The lower triangular matrix U given in* (3.1) *may be decomposed as*

$$
\begin{aligned}
U &= \mathrm{Diag}(1, b_{12}, b_{32}, \ldots, b_{n2})\mathrm{Diag}(I_2, L_{n-2})K_n \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & b_{12} & 0 & 0 & \cdots & 0 \\ 0 & 0 & b_{32} & 0 & \cdots & 0 \\ 0 & 0 & 0 & b_{42} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n2} \end{pmatrix} \cdot \left(\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 1 & \cdots & 1 \end{array}\right) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.
\end{aligned}
$$

Its inverse has a similar decomposition

$$
\begin{aligned}
U^{-1} &= K_n^{-1}\mathrm{Diag}(I_2, L_{n-2}^{-1})\mathrm{Diag}(1, b_{12}^{-1}, b_{32}^{-1}, \ldots, b_{n2}^{-1}) \\
&= L_n\mathrm{Diag}(I_2, K_{n-2})\mathrm{Diag}(1, b_{12}^{-1}, b_{32}^{-1}, \ldots, b_{n2}^{-1}) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \cdot \left(\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{12}^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{32}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{42}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{n2}^{-1} \end{pmatrix}.
\end{aligned}
$$

Next we define the matrix $U_{n,k} : (n-k+2) \times (n-k+2)$, $k = 2, \ldots, n$,

$$
U_{n,k} = e_1e_1' + b_{1k}(e_2e_1' - e_2e_2') + \sum_{i=3}^{n} b_{i+k-2,k}(e_ie_i' - e_ie_2'). \tag{3.5}
$$

The inverse of $U_{n,k}$ is given by

$$
U_{n,k}^{-1} = \sum_{i=1}^{n} e_ie_1' - b_{1k}^{-1}\sum_{i=2}^{n} e_ie_2' + \sum_{i=3}^{n} b_{i+k-2,k}^{-1}e_ie_i'. \tag{3.6}
$$

Note, that $U_{n,2} = U$, and thus $U_{n,k}$ is a direct generalization of U. For example,

$$U_{5,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_{13} & -b_{13} & 0 & 0 \\ 0 & -b_{43} & b_{43} & 0 \\ 0 & -b_{53} & 0 & b_{53} \end{pmatrix}, \quad U_{5,4} = \begin{pmatrix} 1 & 0 & 0 \\ b_{14} & -b_{14} & 0 \\ 0 & -b_{54} & b_{54} \end{pmatrix},$$

$$U_{5,3}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -b_{13}^{-1} & 0 & 0 \\ 1 & -b_{13}^{-1} & b_{43}^{-1} & 0 \\ 1 & -b_{13}^{-1} & 0 & b_{53}^{-1} \end{pmatrix}, \quad U_{5,4}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -b_{14}^{-1} & 0 \\ 1 & -b_{14}^{-1} & b_{54}^{-1} \end{pmatrix}.$$

It has been shown that in order to triangularize $B = B_{n,2}$ the matrices U and U^{-1} could be used. Furthermore, the matrix $U_{n,k}$ and its inverse $U_{n,k}^{-1}$, given in (3.5) and (3.6), respectively, triangularize $B_{n,k}$. Hence, we may state the following interesting result.

Theorem 3.4 *Let* $D_{l,m} = \text{Diag}(I_{l-2+m}, U_{n,l+m})$*, and* $U_{n,k}$ *is defined in (3.5). For* $B_{n,k}$*, given in Definition 1.2, the following relations hold (* $*$ *indicates an unspecified element):*

(i) $U_{n,k} B_{n,k} U_{n,k}^{-1} = \left(\begin{array}{c|c} n-k+1 & * \\ \hline 0 & B_{n,k+1} \end{array}\right)$,

(ii) $\prod_{i=0}^{h} D_{k,h-i} B_{n,k} \prod_{i=0}^{h} D_{k,i}^{-1} = \left(\begin{array}{c|c|c|c} n-k+1 & * & * & * \\ \hline 0 & \ddots & * & * \\ \hline 0 & 0 & n-h-1 & * \\ \hline 0 & 0 & 0 & B_{n,k+h+1} \end{array}\right)$,

(iii) $\prod_{i=0}^{n-2} D_{2,n-2-i} B_{n,2} \prod_{i=0}^{n-2} D_{2,i}^{-1} = \left(\begin{array}{c|c|c|c} n-1 & * & * & * \\ \hline 0 & \ddots & * & * \\ \hline 0 & 0 & 1 & -1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$.

In particular, Theorem 3.4 is important since it shows a way of how to triangularize any $B \in \mathbb{B}_n$. This will be further exploited in the next section.

Theorem 3.5 *Let* $B_{n,k}$ *be given in Definition 1.2. Then* $B_{n,k}$ *has the nonnegative integer eigenvalues* $\{0, 1, \ldots, n-k+1\}$.

4 Triangular Factorization

In this section Theorem 3.4 is explored and we are going to factorize $B \in \mathbb{B}_n$ into a product of three triangular matrices, which will be called a VTU-decomposition. For notational convenience it is assumed that $\prod_{i=h}^{n} a_i = 1$, $\sum_{i=h}^{n} a_i = 0$, if $n < h$, and $\prod_{\substack{i=2\\ i\neq 2}}^{2} a_i = 1$. This convention will be applied throughout the rest of the paper. Indeed, it implies that Theorem 2.1 is true for $n = 1$. The next matrix will later appear as one of the matrices in the triangular VTU-decomposition of $B \in \mathbb{B}_n$:

$$
\begin{aligned}
(4.1)\quad U_n &= e_1e_1' - b_{12}e_2e_2' + \sum_{i=2}^{n}\prod_{m=2}^{i} b_{1m}e_ie_1' - \sum_{j=2}^{n}\sum_{\substack{i=j\\ j\neq 2}}^{n} b_{1j}\prod_{\substack{m=2\\ m\neq j}}^{i} b_{jm}e_ie_j' \\
&= \sum_{i=1}^{n}\prod_{m=2}^{n} b_{1m}e_ie_1' - \sum_{j=2}^{n}\sum_{i=j}^{n} b_{1j}\prod_{\substack{m=2\\ m\neq j}}^{i} b_{jm}e_ie_j',
\end{aligned}
$$

where as before e_i is a standard unit vector of size n.

Theorem 4.1 *Let U_n be given by* (4.1)*, and let $V_n = U_n^{-1}$. Then,*

$$
\begin{aligned}
(4.2)\qquad V_n &= \sum_{k=1}^{n} e_ke_1' - b_{12}^{-1}e_2e_2' + \sum_{k=3}^{n} b_{2k}b_{1k}^{-1}b_{k2}^{-1}e_ke_2' \\
&\quad - \sum_{k=3}^{n} b_{1k}^{-1}\prod_{m=2}^{k-1} b_{km}^{-1}e_ke_k' - \sum_{k=4}^{n}\sum_{l=3}^{k-1} b_{l1}b_{1l}^{-1}\prod_{m=1}^{l-1} b_{km}^{-1}e_ke_l' \\
&= \sum_{k=1}^{n} e_ke_1' - \sum_{k=2}^{n}\sum_{l=2}^{k} b_{l1}b_{1l}^{-1}\prod_{m=1}^{l-1} b_{km}^{-1}e_ke_l'.
\end{aligned}
$$

Proof: We prove the theorem via induction. Taking into account the structure of the matrix U_n the following notation is used:

$$U_n = \begin{pmatrix} U_{n-1} & 0 \\ U_n^{21} & U_n^{22} \end{pmatrix} \qquad \begin{pmatrix} n-1 \times n-1 & n-1 \times 1 \\ 1 \times n-1 & 1 \times 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ U_n^{21} & 0 \end{pmatrix} = \prod_{m=2}^{n} b_{1m} e_n e_1' - \sum_{j=2}^{n-1} b_{1j} \prod_{\substack{m=2 \\ m \neq j}}^{n} b_{jm} e_n e_j', \tag{4.3}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & U_n^{22} \end{pmatrix} = -b_{1n} \prod_{m=2}^{n-1} b_{nm} e_n e_n', \tag{4.4}$$

$$V_n = \begin{pmatrix} V_{n-1} & 0 \\ V_n^{21} & V_n^{22} \end{pmatrix} \qquad \begin{pmatrix} n-1 \times n-1 & n-1 \times 1 \\ 1 \times n-1 & 1 \times 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ V_n^{21} & 0 \end{pmatrix} = e_n e_1' + b_{2n} b_{1n}^{-1} b_{n2}^{-1} e_n e_2' - \sum_{l=3}^{n-1} b_{l1} b_{1l}^{-1} \prod_{m=1}^{l-1} b_{nm}^{-1} e_n e_l', \tag{4.5}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & V_n^{22} \end{pmatrix} = -b_{1n}^{-1} \prod_{m=2}^{n-1} b_{nm}^{-1} e_n e_n'. \tag{4.6}$$

Thus,

$$U_n V_n = \begin{pmatrix} U_{n-1} & 0 \\ U_n^{21} & U_n^{22} \end{pmatrix} \begin{pmatrix} V_{n-1} & 0 \\ V_n^{21} & V_n^{22} \end{pmatrix} = \begin{pmatrix} U_{n-1} V_{n-1} & 0 \\ U_n^{21} V_{n-1} + U_n^{22} V_n^{21} & U_n^{22} V_n^{22} \end{pmatrix}.$$

For $n = 2$ the theorem is obviously true. As induction assumption $U_{n-1} V_{n-1} = I_{n-1}$ will be used and therefore we have to show that $U_n^{21} V_{n-1} + U_n^{22} V_n^{21} = 0$ as well as $U_n^{22} V_n^{22} = 1$, where the last statement immediately follows from (4.4) and (4.6). Now we show that $U_n^{21} V_{n-1} + U_n^{22} V_n^{21} = 0$ and the proof will be decomposed into three parts, i.e. we consider when the first element in $U_n^{21} V_{n-1} + U_n^{22} V_n^{21}$ equals 0, the second equals 0 and elements 3 to n-1 equal 0. The first element in $U_n^{21} V_{n-1} + U_n^{22} V_n^{21}$ equals

$$\prod_{m=2}^{n} b_{1m} - \sum_{j=2}^{n-1} b_{1j} \prod_{m=2}^{n} b_{jm} - b_{1n} \prod_{m=2}^{n-1} b_{nm}. \tag{4.7}$$

By induction assumption it follows from (4.7) that

$$\prod_{m=2}^{n-1} b_{1m} = \sum_{j=2}^{n-1} b_{1j} \prod_{\substack{m=2 \\ m \neq j}}^{n-1} b_{jm}. \tag{4.8}$$

Inserting (4.8) in (4.7) gives

$$\sum_{j=2}^{n-1} b_{1j} \prod_{\substack{m=2 \\ m\neq j}}^{n-1} b_{jm}(b_{1n} - b_{jn}) - b_{1n} \prod_{m=2}^{n-1} b_{nm} = b_{1n}(\sum_{j=2}^{n-1} b_{nj} \prod_{\substack{m=2 \\ m\neq j}}^{n-1} b_{jm} - \prod_{m=2}^{n-1} b_{nm}) = 0,$$

where (4.3) has been applied to $b_{1n} - b_{jn}$. The last equality holds, because of symmetry b_{1j} can be replaced by b_{nj}.

Turning to the second element in $U_n^{21}V_{n-1} + U_n^{22}V_n^{21}$ we have

$$\prod_{m=3}^{n} b_{2m} - \sum_{j=3}^{n-1} b_{1j} \prod_{\substack{m=2 \\ m\neq j}}^{n} b_{jm} b_{2j} b_{1j}^{-1} b_{j2}^{-1} - b_{1n} \prod_{m=2}^{n-1} b_{nm} b_{2n} b_{1n}^{-1} b_{n2}^{-1}$$

$$= \textstyle\prod_{m=3}^{n} b_{2m} - \sum_{j=3}^{n-1} b_{2j} \prod_{\substack{m=3 \\ m\neq j}}^{n} b_{jm} - b_{2n} \prod_{m=3}^{n-1} b_{nm}. \tag{4.9}$$

However, this expression is of the same form as (4.7) and by coping the proof of showing that (4.7) equals 0, we may claim that (4.9) is also identical to 0. Finally we will consider the elements from 3 to n-1 in $U_n^{21}V_{n-1} + U_n^{22}V_n^{21}$:

$$\sum_{j=3}^{n-1} b_{1j} \prod_{\substack{m=2 \\ m\neq j}}^{n} b_{jm} b_{1j}^{-1} \prod_{m=2}^{j-1} b_{jm}^{-1} e_j' + \sum_{j=4}^{n-1} b_{1j} \prod_{\substack{m=3 \\ m\neq j}}^{n} b_{jm} \sum_{l=3}^{j-1} b_{l1} b_{1l}^{-1} \prod_{m=1}^{l-1} b_{jm}^{-1} e_l'$$

$$+ b_{1n} \prod_{m=2}^{n-1} b_{nm} \sum_{l=3}^{n-1} b_{l1} b_{1l}^{-1} \prod_{m=1}^{l-1} b_{nm}^{-1} e_l'. \tag{4.10}$$

Observe that (4.10) is a vector of size n but this is immaterial. We are only interested in the elements 3 to n-1 and the others are authomatically 0. Expression (4.10) can be simplified and after some manipulations, we have

$$\sum_{l=3}^{n-1} \prod_{m=l+1}^{n} b_{lm} e_l' + \sum_{l=3}^{n-2} \sum_{j=l+1}^{n-1} b_{1j} b_{l1} b_{1l}^{-1} b_{j1} \prod_{\substack{m=l \\ m\neq j}}^{n} b_{jm} e_l'$$

$$+ b_{1n} \sum_{l=3}^{n-1} b_{l1} b_{1l}^{-1} b_{n1}^{-1} \prod_{m=l}^{n-1} b_{nm} e_l'. \tag{4.11}$$

However, by Theorem 2.2 (ii), $b_{1j} b_{l1} b_{1l}^{-1} b_{j1}^{-1} = -b_{lj} b_{jl}^{-1}$ and thus (4.11) is identical to

$$\sum_{l=3}^{n-1} \prod_{m=l+1}^{n} b_{lm} e_l' - \sum_{l=3}^{n-2} \sum_{j=l+1}^{n-1} b_{lj} \prod_{\substack{m=l+1 \\ m\neq j}}^{n} b_{jm} e_l' - \sum_{l=3}^{n-2} b_{ln} \prod_{m=l+1}^{n-1} b_{nm} e_l' - b_{n-1n} e_{n-1}'. \tag{4.12}$$

For elements $3, 4, \ldots, n-2$ expression (4.12) is of the same form as (4.7) and hence equals 0. For element $n-1$ expression (4.12) is trivially 0. Thus we have established that $U_n^{21} V_{n-1} + U_n^{22} V_n^{21} = 0$ and hereby the theorem is verified. □

It is interesting to observe how Theorem 2.2 was used in the proof because it shows that the structure of the elements in $B \in \mathbb{B}_n$ is necessary for the result to hold.

Let

$$I_{\{j>k\}} = \begin{cases} 1, & \text{if } j > k, \\ 0, & \text{otherwise.} \end{cases}$$

In the next we spell out the elements of U_n and V_n from the previous theorem.

Theorem 4.2 *Let $U_n = (u_{ij})$ and $V_n = (v_{ij})$ be given by (4.1) and (4.2), respectively. Then*

$$u_{ij} = (-b_{1j})^{I_{\{j>1\}}} \prod_{\substack{k=2 \\ k \neq j}}^{i} b_{jk}, \; i \geq j \tag{4.13}$$

and

$$v_{ij} = \left(-\frac{b_{j1}}{b_{1j}}\right)^{I_{\{j>1\}}} \prod_{k=1}^{j-1} b_{ik}^{-1}, \; i \geq j. \tag{4.14}$$

Example 4.1. For $n = 4$ the matrices U_4 and V_4 are given by

$$U_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_{12} & -b_{12} & 0 & 0 \\ b_{12}b_{13} & -b_{12}b_{23} & -b_{13}b_{32} & 0 \\ b_{12}b_{13}b_{14} & -b_{12}b_{23}b_{24} & -b_{13}b_{32}b_{34} & -b_{14}b_{42}b_{43} \end{pmatrix},$$

$$V_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -b_{12}^{-1} & 0 & 0 \\ 1 & b_{23}/(b_{13}b_{32}) & -1/(b_{13}b_{32}) & 0 \\ 1 & b_{24}/(b_{14}b_{42}) & -b_{31}/(b_{13}b_{41}b_{42}) & -1/(b_{14}b_{42}b_{43}) \end{pmatrix}.$$

We may also relate U_n and V_n to Theorem 3.4.

Theorem 4.3 *Let U_n and V_n be given by (4.1) and (4.2), respectively. Then,*

$$U_n = \prod_{i=0}^{n-2} \mathrm{Diag}(I_{n-i-2}, U_{n,n-i}), \tag{4.15}$$

$$V_n = \prod_{i=0}^{n-2} \mathrm{Diag}(I_i, U_{n,2+i}), \tag{4.16}$$

where $U_{n,k}$ is defined in (3.5).

Before considering the VTU-decomposition, i.e. the factorization $U_nBV_n = T_n$, where T_n is a triangular matrix specified in Theorem 4.4, we show a technical lemma which presents another basic property of $B \in \mathbb{B}_n$.

Lemma 4.1 *Let $B \in \mathbb{B}_n$ and $(U_n^{21} : U_n^{22})$ be the last row in U_n, given in* (4.1). *Then*

$$(U_n^{21} : U_n^{22})B = 0.$$

Proof: Note that U_n^{21} and U_n^{22} were used in the proof of Theorem 4.1. Moreover, as in the proof of that theorem we will divide the present proof into three parts, i.e. here we will consider the first element in $(U_n^{21} : U_n^{22})B$, the elements 2 to $n-1$ and the nth element. For the first element in $(U_n^{21} : U_n^{22})B$ we have

$$\prod_{m=2}^{n} b_{1m} \sum_{j=2}^{n} b_{j1} - \sum_{j=2}^{n-1} b_{1j} \prod_{\substack{m=2\\m\neq j}}^{n} b_{jm}b_{j1} - b_{1n} \prod_{m=2}^{n-1} b_{nm}b_{n1}. \tag{4.17}$$

Pure calculations yield that (4.17) equals 0, for $n = 3, 4$. Now assume that (4.17) is 0 for $n-1$. Expression (4.17) equals

$$\prod_{m=2}^{n-1} b_{1m} \sum_{j=2}^{n-1} b_{j1}b_{1n} + \prod_{m=2}^{n} b_{1m}b_{n1} - \sum_{j=2}^{n} b_{1j} \prod_{\substack{m=2\\m\neq j}}^{n} b_{jm}b_{j1},$$

which via the induction assumption can be written

$$\sum_{j=2}^{n-1} b_{j1} \prod_{\substack{m=2\\m\neq j}}^{n-1} b_{jm}b_{j1}b_{1n} + \prod_{m=2}^{n} b_{1m}b_{n1} - \sum_{j=2}^{n-1} b_{1j} \prod_{\substack{m=2\\m\neq j}}^{n-1} b_{jm}b_{jn}b_{j1} - b_{1n} \prod_{m=2}^{n-1} b_{nm}b_{n1}$$

$$= \sum_{j=2}^{n-1} b_{j1} \prod_{\substack{m=2\\m\neq j}}^{n-1} b_{jm}b_{j1}(b_{1n} - b_{jn}) + \prod_{m=2}^{n} b_{1m}b_{n1} - b_{1n} \prod_{m=2}^{n-1} b_{nm}b_{n1}. \tag{4.18}$$

If we use that $b_{1n} - b_{jn} = b_{nj}b_{1n}b_{1j}^{-1}$ and $(b_{j1} - b_{jn})b_{n1} = b_{j1}b_{nj}$, it can be shown that (4.18) implies

$$\sum_{j=2}^{n-1}\sum_{m=2}^{n-1} b_{jm}b_{j1}b_{n1}b_{1n} + \prod_{m=2}^{n-1} b_{1m}b_{1n}b_{n1} - \sum_{j=2}^{n-1}\prod_{\substack{m=2\\m\neq j}}^{n-1} b_{jm}b_{jn}b_{n1}b_{1n} - \prod_{m=2}^{n-1} b_{nm}b_{n1}b_{1n}$$
$$= \sum_{j=1}^{n-1}\prod_{\substack{m=1\\m\neq j}}^{n-1} b_{jm}b_{n1}b_{1n} - \sum_{j=2}^{n}\prod_{\substack{m=2\\m\neq j}}^{n-1} b_{km}b_{n1}b_{1n} = 0,$$

where the last equality follows from Theorem 2.1 and Corollary 2.2.

We shall show that element $l \in \{2, \ldots, n-1\}$ in $(U_n^{21} : U_n^{22})B$ equals 0, i.e.

$$\prod_{m=2}^{n} b_{1m}b_{1l} - b_{1l}\prod_{\substack{m=2\\m\neq l}}^{n} b_{lm}\sum_{\substack{j=1\\j\neq l}}^{n} b_{jl} - \prod_{\substack{j=2\\j\neq l}}^{n} b_{1j}\prod_{\substack{m=2\\m\neq j}}^{n} b_{jm}b_{jl} = 0. \tag{4.19}$$

It follows after some manipulations that (4.19) equals 0 for $n = 3, 4$. Now, assume that

$$\prod_{m=2}^{n-1} b_{1m}b_{1l} - b_{1l}\prod_{\substack{m=2\\m\neq l}}^{n-1} b_{lm}\sum_{\substack{j=1\\j\neq l}}^{n-1} b_{jl} - \sum_{\substack{j=2\\j\neq l}}^{n-1} b_{1j}\prod_{\substack{m=2\\m\neq j}}^{n-1} b_{jm}b_{jl} = 0. \tag{4.20}$$

By symmetry it follows from (4.20) that by replacing 1 by n

$$\prod_{m=2}^{n-1} b_{nm}b_{nl} - b_{nl}\prod_{\substack{m=2\\m\neq l}}^{n-1} b_{lm}\sum_{\substack{j=2\\j\neq l}}^{n} b_{jl} - \sum_{\substack{j=2\\j\neq l}}^{n-1} b_{nj}\prod_{\substack{m=2\\m\neq j}}^{n-1} b_{jm}b_{jl} = 0, \tag{4.21}$$

which will be used later. Applying (4.20) in (4.19) gives

$$b_{1l}\prod_{\substack{m=2\\m\neq l}}^{n-1} b_{lm}\sum_{\substack{j=1\\j\neq l}}^{n-1} b_{jl}b_{1n} + \sum_{\substack{j=2\\j\neq l}}^{n-1} b_{1j}\prod_{m=2}^{n-1} b_{jm}b_{jl}b_{1n} - b_{1l}\prod_{\substack{m=2\\m\neq l}}^{n} b_{lm}\sum_{\substack{j=1\\j\neq l}}^{n} b_{jl} - \sum_{\substack{j=2\\j\neq l}}^{n} b_{1j}\prod_{\substack{m=2\\m\neq j}}^{n} b_{jm}b_{jl}$$
$$=\sum_{\substack{j=1\\j\neq l}}^{n-1} b_{1l}\prod_{\substack{m=2\\m\neq j}}^{n-1} b_{lm}b_{jl}(b_{1n}-b_{ln}) + \sum_{j=2}^{n-1} b_{1j}\prod_{\substack{m=2\\m\neq j}}^{n} b_{jm}b_{jl}(b_{1n}-b_{jn}) - b_{1l}\prod_{\substack{m=2\\m\neq l}}^{n} b_{lm}b_{nl} - b_{1n}\prod_{\substack{m=2\\m\neq n\\l\neq n}}^{n} b_{nm}b_{nl}. \tag{4.22}$$

Using that $(b_{1n} - b_{ln}) = b_{nl} b_{1n} b_{1l}^{-1}$, $(b_{1n} - b_{jn}) = b_{nj} b_{1n} b_{1j}^{-1}$ and performing some more calculations leads to that (4.22) is identical to

$$b_{1n}\{\sum_{\substack{j=2\\ j\neq l}}^{n} \prod_{\substack{m=2\\ m\neq l}}^{n-1} b_{lm} b_{jl} b_{nl} + \sum_{\substack{k=2\\ k\neq l}}^{n-1} \prod_{\substack{m=2\\ m\neq k}}^{n-1} b_{km} b_{kl} b_{nk} - \prod_{\substack{m=2\\ m\neq l}}^{n-1} b_{nm} b_{nl}\},$$

which by (4.21) equals 0. Thus, the lth element in $(U_n^{21} : U_n^{22})B$, where $l \in \{2, \ldots, n-1\}$, equals 0 and it remains to show that the last element in $(U_n^{21} : U_n^{22})B$ is 0. This means that we have to show that

$$\prod_{m=2}^{n} b_{1m} b_{1n} - b_{1n} \prod_{m=2}^{n-1} b_{nm} \sum_{j=1}^{n-1} b_{jn} - \sum_{j=2}^{n-1} b_{1j} \prod_{\substack{m=2\\ m\neq j}}^{n} b_{jm} b_{jn} = 0. \tag{4.23}$$

By straightforward manipulations it follows that (4.23) holds for $n = 3, 4$. Now, assume that (4.23) is true for $n-1$, i.e.

$$\prod_{m=2}^{n-1} b_{1m} b_{1n-1} - b_{1n-1} \prod_{m=2}^{n-2} b_{n-1m} \sum_{j=1}^{n-1} b_{jn-1} - \sum_{j=2}^{n-2} \prod_{\substack{m=2\\ m\neq j}}^{n-1} b_{jm} b_{jn-1} = 0,$$

which implies

$$b_{1n-1}\{\prod_{m=2}^{n-1} b_{lm} + \sum_{j=2}^{n-2} \prod_{\substack{m=2\\ m\neq j}}^{n-1} b_{jm} b_{j1} - \prod_{m=2}^{n-2} b_{n-1m} \sum_{j=1}^{n-2} b_{jn-1} - \sum_{j=2}^{n-2} \prod_{\substack{m=2\\ m\neq j}}^{n-1} b_{jm} b_{jn-1}\} = 0. \tag{4.24}$$

By changing indices, $1 \to 2, 2 \to 3, \ldots, n-1 \to n, n \to 1$ we obtain from (4.24)

$$\prod_{m=3}^{n} b_{2m} + \sum_{j=3}^{n-1} \prod_{\substack{m=3\\ m\neq j}}^{n} b_{jm} b_{j2} - \prod_{m=3}^{n-1} b_{nm} \sum_{j=2}^{n-1} b_{jn} - \sum_{j=3}^{n-1} \prod_{m=3}^{n} b_{jm} b_{jn} = 0. \tag{4.25}$$

Returning to (4.23) we have that the left hand side equals

$$b_{1n}\{\prod_{m=2}^{n} b_{1m} - \prod_{m=2}^{n-1} b_{nm} \sum_{j=1}^{n-1} b_{jn} - \sum_{j=2}^{n-1} b_{jn} \prod_{\substack{m=2\\ m\neq j}}^{n} b_{jm} + \sum_{j=2}^{n-1} b_{j1} \prod_{\substack{m=2\\ m\neq j}}^{n} b_{jm}\}$$

$$= b_{1n}\{1 - \prod_{m=1}^{n-1} b_{nm} - \prod_{m=2}^{n-1} b_{nm} \sum_{j=1}^{n-1} b_{jn} - \sum_{j=2}^{n-1} b_{jn} \prod_{\substack{m=2\\ m\neq j}}^{n} b_{jm}\}, \tag{4.26}$$

where Corollary 2.1 has been applied. Reorganizing terms in (4.26), where in particular it is used that $b_{n1}+b_{1n}=1$ and $b_{jn}=1-b_{nj}$, we obtain that (4.26) is identical to

$$b_{1n}\{1-\prod_{m=2}^{n-1} b_{nm}-\prod_{m=2}^{n-1} b_{nm}\sum_{j=2}^{n-1} b_{jn}-\sum_{j=2}^{n-1} b_{jn}\prod_{\substack{m=2\\ m\neq j}}^{n} b_{jm}\}$$

$$=b_{1n}\{1-\prod_{m=2}^{n-1} b_{nm}-\prod_{m=2}^{n-1} b_{nm}\sum_{j=2}^{n-1} b_{jn}-\sum_{j=2}^{n-1} b_{jn}\prod_{\substack{m=2\\ m\neq j}}^{n-1} b_{jm}+\sum_{j=2}^{n-1} b_{jn}\prod_{\substack{m=2\\ m\neq j}}^{n-1} b_{jm}b_{nj}\}$$

$$=b_{1n}\{\sum_{j=2}^{n-1} b_{jn}\prod_{\substack{m=2\\ m\neq j}}^{n-1} b_{jm}b_{nj}-\prod_{m=2}^{n-1} b_{nm}\sum_{j=2}^{n-1} b_{jn}\}, \tag{4.27}$$

where for the last equality Corollary 2.2 has been applied to

$$\prod_{m=2}^{n-1} b_{nm}+\sum_{j=2}^{n-1} b_{jn}\prod_{\substack{m=2\\ m\neq j}}^{n-1} b_{jm}=\sum_{j=2}^{n}\prod_{m=2}^{n} b_{jm}.$$

Since $b_{j2}b_{nj}=b_{n2}(b_{j2}-b_{jn})$ it follows that the right hand side in (4.27) can be written

$$b_{1n}b_{n2}\{\sum_{j=3}^{n-1}\prod_{\substack{m=3\\ m\neq j}}^{n} b_{jm}b_{j2}+\prod_{m=3}^{n} b_{2m}-\prod_{m=3}^{n-1} b_{nm}\sum_{j=2}^{n-1} b_{jn}-\sum_{j=3}^{n-1}\prod_{\substack{m=3\\ m\neq j}}^{n-1} b_{jm}b_{jn}\} \tag{4.28}$$

and finally from (4.25) it follows that (4.28) equals 0. Thus the proof of the lemma is completed. □

Now one of the main theorems of the paper is formulated.

Theorem 4.4 *(VTU-decomposition) Let $B\in\mathbb{B}_n$, U_n and $V_n=U_n^{-1}$ be the triangular matrices given by (4.1) and (4.2), respectively. Then $U_nBV_n=T_n$, where the upper triangular T_n equals*

$$\sum_{k=1}^{n}(n-k)e_ke_k'+\sum_{r=3}^{n}\sum_{k=1}^{r-2}\sum_{l=k+1}^{r-1}\prod_{m=k+1}^{l} b_{rm}^{-1}b_{lr}e_ke_l'-\sum_{r=3}^{n}\sum_{k=1}^{r-2}\prod_{m=k+1}^{r-1} b_{rm}^{-1}e_ke_r'-\sum_{k=1}^{n-1} e_ke_{k+1}'.$$

Proof: It can be shown, via direct calculations, that the theorem is true for, say $n<5$. After the proof we show some of them for $n=3$. Suppose now that $U_{n-1}B_{n-1}V_{n-1} = T_{n-1}$ holds, where $B_{n-1} \in \mathbb{B}_{n-1}$. Using the notation of Theorem 4.1 we have

$$U_nBV_n = \begin{pmatrix} U_{n-1} & 0 \\ U_n^{21} & U_n^{22} \end{pmatrix} B \begin{pmatrix} V_{n-1} & 0 \\ V_n^{21} & V_n^{22} \end{pmatrix}$$

and in the subsequent B is partitioned as

$$B = \begin{pmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{pmatrix} \qquad \begin{pmatrix} n-1 \times n-1 & n-1 \times 1 \\ 1 \times n-1 & 1 \times 1 \end{pmatrix}.$$

From Lemma 4.1 it follows that $(U_n^{21} : U_n^{22})B = 0$ and thus

$$(4.29) \quad U_nBV_n = \begin{pmatrix} U_{n-1}B^{11}V_{n-1} + U_{n-1}B^{12}V_n^{21} & U_{n-1}B^{12}V_n^{22} \\ 0 & 0 \end{pmatrix}.$$

The partition in (4.29) will be studied in some detail. First $U_{n-1}B^{12}V_n^{22}$ is considered, i.e. we are going to show that $U_{n-1}B^{12}V_n^{22}$ equals the first $n-1$ elements in the nth column of T_n. Let $T = (t_{ij})$, where $t_{ij} = 0$, if $i > j$. Now for U_{n-1}, V_n and B we have

$$(4.30) \qquad V_n^{22} = -b_{1n}^{-1} \prod_{m=2}^{n-1} b_{nm}^{-1},$$

$$(4.31) \quad U_{n-1}B^{12} = b_{1n}e_1 - b_{12}b_{2n}e_2 + \sum_{i=2}^{n-1} \prod_{m=2}^{i} b_{1m}b_{1n}d_i - \sum_{j=2}^{n-1} \sum_{\substack{i=j \\ j\neq 2}}^{n-1} b_{ij}b_{jn} \prod_{\substack{m=2 \\ m\neq j}}^{i} b_{jm}d_i,$$

where d_i is the unit vector of size $n-1$.

The first element in $U_{n-1}B^{12}$ equals b_{1n}. Thus, the first element in $U_{n-1}B^{12}V_n^{22}$ equals $b_{1n}V_n^{22}$. However, from (4.30) and the statement about T_n we may conclude that $b_{1n}V_n^{22} = t_{1n}$.

For the second element, i.e. t_{2n}, it follows from (4.31):

$$-(-b_{12}b_{2n} + b_{12}b_{1n})b_{1n}^{-1} \prod_{m=2}^{n-1} b_{nm}^{-1} = -b_{n2}b_{1n}b_{1n}^{-1} \prod_{m=2}^{n-1} b_{nm}^{-1} = -\prod_{m=3}^{n-1} b_{nm}^{-1},$$

which equals t_{2n}.

For elements 3 to $n-1$ in $U_{n-1}B^{12}V_n^{22}$ we apply Corollary 2.2, and obtain

$$\begin{aligned}
&\{\sum_{i=3}^{n-1}\prod_{m=2}^{i} b_{1m}b_{1n}d_i - \sum_{j=2}^{n-1}\sum_{i=j}^{n-1} b_{1j}b_{jn}\prod_{\substack{m=2\\ m\neq j}}^{i} b_{jm}d_i\}V_n^{22} \\
= \; & \{\sum_{i=3}^{n-1} b_{1n}\prod_{m=2}^{i} b_{1m}d_i\} - \sum_{i=3}^{n-1}\sum_{j=2}^{i} b_{1j}b_{jn}\prod_{\substack{m=2\\ m\neq j}}^{i} b_{jm}d_i\}V_n^{22} \\
= \; & (\sum_{i=3}^{n-1} b_{1n}\{1-\sum_{j=2}^{i} b_{jn}\prod_{\substack{m=2\\ m\neq j}}^{i} b_{jm}\}d_i)V_n^{22} = \sum_{i=3}^{n-1} b_{1n}\prod_{m=2}^{i} b_{nm}d_iV_n^{22},
\end{aligned}$$

which equals $\sum_{i=3}^{n-1} t_{in}d_i$. Thus, t_{in}, $3 \le i \le n-1$ has been obtained.

Now $U_{n-1}B^{11}V_{n-1} + U_{n-1}B^{12}V_n^{21}$, given in (4.29), will be studied. This expression equals

$$U_{n-1}B_{n-1}V_{n-1} + I - \sum_{i=1}^{n-1} b_{in}U_{n-1}d_id_i'V_{n-1} + \sum_{i=1}^{n-1} U_{n-1}b_{in}d_iV_n^{21} \tag{4.32}$$

and the two last terms in (4.32) will be exploited. It will be shown that the sum forms an upper triangular matrix and then via (4.31) we obtain a fairly useful recursive relation between U_nBV_n and $U_{n-1}B_{n-1}V_{n-1}$. By performing a number of calculations we obtain that

$$\begin{aligned}
\sum_{i=1}^{n-1} U_{n-1}b_{in}d_iV_n^{21} = {} & (b_{1n}d_1 + b_{n2}b_{1n}d_2 + \sum_{k=3}^{n-1}\prod_{m=2}^{k} b_{nm}b_{1n}d_k) \\
& \times(d_1' + b_{2n}b_{1n}^{-1}b_{n2}^{-1}d_2' - \sum_{l=3}^{n-1} b_{l1}b_{1l}^{-1}\prod_{m=1}^{l-1} b_{nm}^{-1}d_l'),
\end{aligned} \tag{4.33}$$

which should be compared to

$$-\sum_{i=1}^{n-1} b_{in}U_{n-1}d_id_i'V_{n-1}. \tag{4.34}$$

For the first column out of the $n-1$ columns in (4.34) we note that it equals

$$-\{b_{1n}d_1 + b_{n2}b_{1n}d_2 + \sum_{k=3}^{n-1}\sum_{j=1}^{k}\prod_{\substack{m=1\\ m\neq j}}^{k} b_{jm}b_{jn}d_k - \sum_{k=3}^{n-1}\sum_{j=2}^{k}\prod_{\substack{m=2\\ m\neq j}}^{k} b_{jm}b_{jn}d_k\},$$

which by Theorem 2.1 can be written

$$-\{b_{1n}d_1 + b_{n2}b_{1n}d_2 + \sum_{k=3}^{n-1} b_{1n} \prod_{m=2}^{k} b_{nm}d_k\}.$$

Thus the sum of the first columns in (4.33) and (4.34) equals 0. Moreover, for the second column in (4.34) we have that below the diagonal it stands

$$-\{\sum_{k=3}^{n-1}\prod_{m=3}^{k} b_{2m}b_{2n}d_k - \sum_{k=3}^{n-1}\sum_{j=3}^{k} b_{jn}b_{2j}b_{j2}^{-1} \prod_{\substack{m=2\\m\neq j}}^{k} b_{jm}d_k\},$$

which equals

$$-\{\sum_{k=3}^{n-1}\prod_{m=3}^{k} b_{nm}b_{2n}d_k\}.$$

This can be verified by applying the same calculations as will be used now when obtaining a convenient expression for the elements below the diagonal in the rth column, $3 \leq r \leq n-1$, in (4.34).

We have

$$-b_{rn}\sum_{k=r+1}^{n-1} b_{1r} \prod_{\substack{m=2\\m\neq r}}^{k} b_{rm}b_{1r}^{-1} \prod_{m=2}^{r-1} b_{rm}^{-1}d_k - \sum_{j=r+1}^{n-1} b_{jn} \sum_{k=j}^{n-1} b_{1j} \prod_{\substack{m=2\\m\neq i}}^{k} b_{jm}b_{r1}b_{1r}^{-1} \prod_{m=1}^{r-1} b_{jm}^{-1}d_k$$

$$= -\sum_{k=r+1}^{n-1}\prod_{m=r+1}^{k} b_{rm}b_{rn}d_k + \sum_{j=r+1}^{n-1}\sum_{j=r+1}^{k}\prod_{\substack{m=r+1\\m\neq j}} b_{jm}b_{rj}b_{jn}d_k$$

$$= -\sum_{k=r+1}^{n-1}\sum_{j=r}^{k}\prod_{\substack{m=r\\m\neq j}}^{k} b_{jm}b_{jn}d_k + \sum_{k=r+1}^{n-1}\sum_{j=r+1}^{k}\prod_{\substack{m=r+1\\m\neq j}}^{k} b_{jm}b_{jn}d_k.$$

However, Theorem 2.1 simplifies this expression and we obtain that it can be written

$$-\sum_{k=r+1}^{n-1}\{(1-b_{nr}b_{nr+1}\times\cdots\times b_{nk})-(1-b_{nr+1}b_{nr+2}\times\cdots\times b_{nk})\}d_k$$

$$= -\sum_{k=r+1}^{n-1} b_{rn}\prod_{m=r+1}^{k} b_{nm}d_k. \tag{4.35}$$

Hence, if we add the elements below the diagonal in (4.33) and (4.34) the sum equals 0.

In the next the diagonal elements in (4.34) are considered. For the first element we have already shown that it equals $-b_{1n}$. For the second element we have $-b_{2n}$ and if we consider the rth diagonal element, $3 \leq r \leq n-1$, we get

$$-b_{rn} b_{1r} \prod_{m=2}^{r-1} b_{rm} b_{1r}^{-1} \prod_{m=2}^{r-1} b_{rm}^{-1} = -b_{rn}. \tag{4.36}$$

Therefore, by utilizing (4.36) it follows that the sum of the diagonal elements in (4.33) and (4.34) is 0. Thus, we have shown that if summing (4.33) and (4.34) in the first and second column all elements on the diagonal and below are 0.

Furthermore, if we in (4.33) for given l and any $k > l$ multiply the terms we obtain

$$\begin{aligned} &-\sum_{k=l+1}^{n-1} \prod_{m=2}^{k} b_{nm} b_{1n} b_{l1} b_{1l}^{-1} \prod_{m=1}^{l-1} b_{nm}^{-1} d_k = -\sum_{k=l+1}^{n-1} \prod_{m=l}^{k} b_{nm} b_{1n} b_{l1} b_{1l}^{-1} b_{n1}^{-1} d_k \\ &= -\sum_{k=l+1}^{n-1} \prod_{m=l+1}^{k} b_{nm} b_{1n} b_{l1} b_{1l}^{-1} b_{n1}^{-1} b_{nl} d_k = \sum_{k=l+1}^{n-1} \prod_{m=l+1}^{k} b_{nm} b_{ln} d_k, \end{aligned} \tag{4.37}$$

since by Theorem 2.2

$$b_{1n} b_{l1} b_{1l}^{-1} b_{n1}^{-1} b_{nl} = b_{1n} b_{l1} b_{ln}^{-1} b_{ln} b_{1l}^{-1} b_{n1}^{-1} b_{nl} = -b_{ln}.$$

Now we may conclude that (4.32) can be written

$$\begin{aligned} &U_{n-1} B_{n-1} V_{n-1} + I + b_{2n} b_{n2}^{-1} d_1 d_2' - \sum_{l=3}^{n-1} b_{1n} b_{l1} b_{1l}^{-1} \prod_{m=1}^{l-1} b_{nm}^{-1} d_1 d_l' \\ &- \sum_{l=3}^{n-1} b_{n2} b_{1n} b_{l1} b_{1l}^{-1} \prod_{m=1}^{n-1} b_{nm}^{-1} d_2 d_l' - \sum_{k=3}^{n-2} \sum_{l=k+1}^{n-1} \prod_{m=2}^{k} b_{nm} b_{1n} b_{l1} b_{1l}^{-1} \prod_{o=1}^{l-1} b_{no}^{-1} d_k d_l'. \end{aligned} \tag{4.38}$$

Using Theorem 2.2 (ii) $b_{1n} b_{l1} b_{1l}^{-1} b_{n1}^{-1} = -b_{ln} b_{nl}^{-1}$, and it follows that (4.38) is identical to

$$U_{n-1} B_{n-1} V_{n-1} + I + \sum_{k=1}^{n-2} \sum_{l=k+1}^{n-1} \prod_{m=k+1}^{l} b_{nm}^{-1} b_{ln} d_k d_l',$$

where $\sum_{k=1}^{n-2} \sum_{l=k+1}^{n-1} \prod_{m=k+1}^{l} b_{nm}^{-1} b_{ln} d_k d_l'$ is strictly upper triangular. Therefore, it follows that if summing (4.33) and (4.34) a strictly upper triangular matrix is obtained, i.e. its diagonal elements equal all 0. Finally by observing the recursiveness of $U_nB_nV_n$, i.e.

$$\begin{aligned} U_nB_nV_n &= (I_{n-1}:0)'U_{n-1}B_{n-1}V_{n-1}(I_{n-1}:0) - \sum_{k=1}^{n-2}\prod_{m=k+1}^{n-1} b_{nm}^{-1}e_ke_n' - e_{n-1}e_n' \\ &+ \sum_{k=1}^{n-1} e_ke_k' + \sum_{k=1}^{n-2}\sum_{l=k+1}^{n-1}\prod_{m=k+1}^{l} b_{nm}^{-1}b_{ln}e_ke_l', \end{aligned}$$

we obtain T_n given in the statement of the theorem. □

Corollary 4.1 *Let $T_n = (t_{ij})$ be the upper triangular matrix defined in Theorem 4.4. Then the elements of T_n are given by*

$$t_{ij} = \sum_{k=j+1}^{n}\prod_{l=i+1}^{j} b_{kl}^{-1} - \sum_{k=i}^{j-1} t_{ik} = \sum_{k=j+1}^{n}\prod_{l=i+1}^{j} b_{kl}^{-1} - I_{\{j>i\}}\sum_{k=j}^{n}\prod_{l=i+1}^{j-1} b_{kl}^{-1}.$$

Observe that the expression implies that $t_{ii} = n-i$. Moreover, $T_n1 = 0$. The structure of the matrix T_n is the following

$$T_n = \begin{pmatrix} n-1 & \sum_{i'=3}^{n} b_{i'2}^{-1}-(n-1) & \sum_{i'=4}^{n}\prod_{j=2}^{3} b_{i'j}^{-1}-\sum_{i'=3}^{n} b_{i'2}^{-1} & \cdots & \cdots & -\prod_{j'=2}^{n-1} b_{nj'}^{-1} \\ 0 & n-2 & \sum_{i'=4}^{n} b_{i'3}^{-1}-(n-2) & \sum_{i'=5}^{n}\prod_{j=3}^{4} b_{i'j}^{-1}-\sum_{i'=4}^{n} b_{i'3}^{-1} & \cdots & -\prod_{j=3}^{n} b_{i'j}^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 2 & b_{nn-1}^{-1}-2 & -b_{nn-1}^{-1} \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This section is ended by showing some detailed calculations for $n = 3$.

EXAMPLE 4.2. For $n = 3$

$$T_3 = \begin{pmatrix} 2 & b_{32}^{-1}-2 & -b_{32}^{-1} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

and $n = 4$

$$T_4 = \begin{pmatrix} 3 & \sum_{i=3}^{4} b_{i2}^{-1}-3 & \prod_{j=2}^{3} b_{4j}^{-1}-\sum_{i=3}^{4} b_{i2}^{-1} & -\prod_{j=2}^{3} b_{4j}^{-1} \\ 0 & 2 & b_{43}^{-1}-2 & -b_{43}^{-1} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From (4.13) and (4.14) in Theorem 4.2 we have

$$U_3 = \begin{pmatrix} 1 & 0 & 0 \\ b_{12} & -b_{12} & 0 \\ b_{12}b_{13} & -b_{12}b_{23} & -b_{13}b_{32} \end{pmatrix}, \qquad V_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -b_{12}^{-1} & 0 \\ 1 & b_{23}b_{13}^{-1}b_{32}^{-1} & -b_{13}^{-1}b_{32}^{-1} \end{pmatrix}.$$

We are going to show that $V_3T_3U_3 = B \in \mathbb{B}_3$. Now

$$\begin{aligned} T_3U_3 &= \begin{pmatrix} 2 + b_{32}^{-1}b_{12} - 2b_{12} - b_{32}^{-1}b_{12}b_{13} & +2b_{12} + b_{32}^{-1}b_{23}b_{12} - b_{12}b_{32}^{-1} & b_{13} \\ b_{12} - b_{12}b_{13} & -b_{12} + b_{12}b_{23} & b_{13}b_{32} \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 - 2b_{12} + (b_{12} - b_{13}) & b_{12}b_{32}^{-1}(1 - b_{23}) + 2b_{12} & b_{13} \\ b_{12}b_{31} & -b_{12}b_{32} & b_{13}b_{32} \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} b_{21} + b_{31} & b_{12} & b_{13} \\ b_{12}b_{31} & -b_{12}b_{32} & b_{13}b_{32} \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} &V_3T_3U_3 \\ &= \begin{pmatrix} b_{21} + b_{31} & b_{12} & b_{13} \\ b_{21} & b_{12} + b_{32} & -b_{13}b_{12}^{-1}b_{32} + b_{13} \\ b_{21} + b_{31} + b_{23}b_{13}^{-1}b_{32}^{-1}b_{12}b_{31} & b_{12} - b_{12}b_{32}b_{23}b_{13}^{-1}b_{32}^{-1} & b_{13} + b_{13}b_{32}b_{23}b_{13}^{-1}b_{32}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} b_{21} + b_{31} & b_{12} & b_{13} \\ b_{21} & b_{12} + b_{32} & -(b_{32} - b_{31}) + b_{13} \\ b_{31} & b_{12} - (b_{23} - b_{21}) & b_{13} + b_{23} \end{pmatrix} \\ &= \begin{pmatrix} b_{21} + b_{31} & b_{12} & b_{13} \\ b_{21} & b_{12} + b_{32} & b_{23} \\ b_{31} & b_{32} & b_{13} + b_{23} \end{pmatrix} = B, \end{aligned}$$

where in the above calculations we have used (1.4) and Theorem 2.2 (ii).

5 Eigenvectors of the matrix B

We already know from Theorem 3.2 that the matrix $B \in \mathbb{B}_n$ has eigenvalues $\{0, 1, \ldots, n-1\}$. This can also be seen from the structure of the matrix T_n given in Corollary 4.1 and the fact that the matrices B and T_n are similar, i.e. $U_nBU_n^{-1} = T_n$. The right eigenvectors of the matrix B are of special interest in sampling theory when B is a function of the inclusion probabilities, outlined in the Introduction.

We are going to present the eigenvectors of the matrix $B \in \mathbb{B}_n$ in a general form.

From Section 2 we know that $U_n B U_n^{-1} = T_n$, where the matrix T_n is an upper-triangular matrix given by Theorem 4.4. Since B and T_n are similar, they have the same eigenvalues and then the eigenvectors of B are rather easy to obtain using the eigenvectors of T_n. In the next theorem we shall obtain explicit expressions for the eigenvectors of the matrix T_n.

Theorem 5.1 *Let T_n be given by Theorem 4.4. Then there exist upper triangular matrices V_T and U_T such that*

$$\begin{aligned} T_n &= U_T \Lambda V_T, \\ \Lambda &= diag(n-1, n-2, \dots, 1, 0), \\ U_T &= V_T^{-1}. \end{aligned} \tag{5.1}$$

The matrix $U_T = (u_{ij})$ is given by

$$u_{ij} = 1 + \sum_{g=1}^{j-i} (-1)^g \sum_{i_1 < \cdots < i_g}^{[j+1,n]} \sum_{j_1 < \cdots < j_g}^{[i+1,j]} \prod_{k=1}^{g} \frac{b_{j_k, i_k}}{b_{i_k, j_k}}, \quad i \le j.$$

The matrix $V_T = (v_{ij})$ satisfies

$$v_{ii} = 1, \; i = 1, \dots, n, \tag{5.2}$$

$$v_{ii+1} = -1 + \sum_{i_1 = i+2}^{n} \frac{b_{i+1 i_1}}{b_{i_1 i+1}}, \tag{5.3}$$

$$v_{ij} = \sum_{h=j-i-1}^{j-i} (-1)^{j-i-h} \sum_{i_1 \le \cdots \le i_h}^{[i+h+1,n]} \prod_{k=i+1}^{i+h} \frac{b_{k, i_{k-i}}}{b_{i_{k-i}, k}}, \quad j - i > 1. \tag{5.4}$$

Proof: We are firstly going to show that $V_T = U_T^{-1}$. Let as in Theorem 2.2 $c_{ij} = b_{ij}^{-1} b_{ji}$ and in the present proof we are often going to refer to this theorem. Since many calculations are rather lengthy, some of them will be omitted. There is certainly place for improvements but this is left for the future.

The basic idea of the proof is to note that

$$U_T = R + Z^U, \tag{5.5}$$

$$V_T = R^{-1} + Z^V, \tag{5.6}$$

where R is a unit upper triangular matrix, i.e. an upper triangular of ones, $Z^U = (z_{ij}^U)$ is a strictly upper triangular matrix given by

$$(5.7) \qquad z_{ij}^U = \sum_{g=1}^{j-i}(-1)^g \sum_{i_1<\cdots<i_g}^{[j+1,n]} \sum_{j_1<\cdots<j_g}^{[i+1,j]} \prod_{k=1}^{g} c_{i_k j_k},$$

and z_{ij}^V in the strictly upper triangular matrix $Z^V = (z_{ij}^V)$ are given by

$$(5.8) \qquad z_{ii+1}^V = \sum_{i_1=i+2}^{n} c_{i_1 i+1},$$

$$(5.9) \qquad z_{ij}^V = \sum_{h=j-i-1}^{j-i} (-1)^{j-i-h} \sum_{i_1\leq\cdots\leq i_h}^{[i+h+1,n]} \prod_{k=i+1}^{i+h} c_{i_{k-i}k}, \; j-i>1.$$

It follows immediately that

$$V_T U_T = (R^{-1} + Z^V)(R + Z^U) = I + R^{-1}Z^U + Z^V R + Z^V Z^U$$

and the aim is to show that $V_T U_T = I$. Therefore, it will be shown that

$$Z^V Z^U = -R^{-1}Z^U - Z^V R,$$

which is equivalent to the interesting equation

$$(5.10) \qquad Z^V R R^{-1} Z^U = -R^{-1}Z^U - Z^V R.$$

Some calculations yield

$$R^{-1}Z^U = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(z_{ij}^U - z_{i+1j}^U)e_i e_j'$$

$$(5.11) \qquad = \sum_{i=1}^{n-2}\sum_{j>i}^{n-1}\sum_{g=1}^{j-i} \sum_{i_1<\cdots<i_g}^{[j+1,n]} \sum_{j_2<\cdots<i_k}^{[i+2,j]} \prod_{k=2}^{g} c_{i_k j_k} c_{i_1 i+1} e_i e_j'$$

and

$$(5.12) \qquad Z^V R = \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} \sum_{i_1\leq\cdots\leq i_{j-i}}^{[j+1,n]} \prod_{k=1}^{j-i} c_{i_k i+k} e_i e_j'.$$

When proceeding we will instead of (5.10) verify that

$$(5.13) \qquad R^{-1}Z^U = (I + Z^V R)^{-1} - I.$$

The reason is that the right hand side of (5.13) is easy to interpret, since $Z^V R$ is strictly upper triangular. The next lemma provides a useful result. The proof is omitted since it follows by straightforward calculations (e.g. see Harville, 1997, for related material).

Lemma 5.1 *Let $T = (t_{ij}) : n \times n$ be any strictly upper triangular matrix. Then*

$$\begin{aligned}\{(I+T)^{-1}\}_{ii} &= 1, \quad i = 1, 2, \ldots, n,\\ \{(I+T)^{-1}\}_{ij} &= -t_{ij} + \sum_{g=2}^{j-i}(-1)^g \sum_{i_1<\cdots<i_{g-1}}^{[i+1,j-1]} t_{ii_1} \prod_{k=2}^{g-1} t_{i_{k-1}i_k} t_{i_{g-1}j}.\end{aligned}$$

It is interesting to observe that the elements in $\{(I+T)^{-1}\}_{ij}$ and $\{(I+T)^{-1}\}_{i+1j+1}$ are of the same form. For example the second expression is obtained from the first if reindexing: $i \to i+1, i_1 \to i_1+1, i_2 \to i_2+1, \ldots, j \to j+1$. Therefore it follows that the most difficult part to verify in (5.13) is to verify that the last elements in the first row in both sides are equal which is the same as showing

$$\begin{aligned}-\sum_{k=3}^{n-2}(R^{-1}Z^U)_{1k}(Z^V R)_{kn-1} \quad -(Z^V R)_{1n-1} + (Z^V R)_{12}(Z^V R)_{2n-1}&\\ = (R^{-1}Z^U)_{1n-1}.& \qquad (5.14)\end{aligned}$$

To verify the equality of the other elements in (5.13) follows by symmetry. To prove (5.14) is a fairly straightforward exercise where the key trick is to use Theorem 2.2. We will only show (5.14) when $n = 7$. The general case follows from some rather tedious calculations which are not of any interest.

From (5.11) and (5.12) it follows that

$$\sum_{k=3}^{5}(R^{-1}Z^U)_{1k}(Z^V R)_{k6} = \sum_{k=3}^{5}\sum_{g=1}^{k-1}(-1)^g \sum_{i_1<\cdots<i_g}^{[k+1,7]} \sum_{j_2<\cdots<j_g}^{[3,7]} \prod_{l=2}^{g} c_{i_l j_l} c_{i_1 2} \prod_{m=1}^{6-k} c_{7k+m}. \qquad (5.15)$$

This expression will now be exploited for $k = 3, 4, 5$ and $g = 1, 2, \ldots, k-1$. For $k = 3$ and $g = 1$ it follows from (5.15) that

$$-\sum_{i_4}^{7} c_{i_1 2} \prod_{m=1}^{3} c_{73+m},$$

has to be considered, which by Theorem 2.2 equals

$$\begin{aligned}&-c_{42}c_{74}c_{75}c_{76}-c_{52}c_{74}c_{75}c_{76}-c_{62}c_{74}c_{75}c_{76}-c_{72}c_{74}c_{75}c_{76}\\ &\quad=c_{72}c_{75}c_{76}+c_{72}c_{74}c_{76}+c_{72}c_{75}c_{75}-c_{72}c_{74}c_{75}c_{76}.\end{aligned}\tag{5.16}$$

For $k=3$ and $g=2$

$$\begin{aligned}&\sum_{i_1=4}^{6}\sum_{i_2=i_1+1}^{7}c_{i_2 3}c_{i_1 2}c_{74}c_{75}c_{76}\\ &\quad=c_{42}c_{53}c_{74}c_{75}c_{76}+c_{42}c_{63}c_{74}c_{75}c_{76}+c_{42}c_{73}c_{74}c_{75}c_{76}\\ &\qquad+c_{52}c_{63}c_{74}c_{75}c_{76}+c_{52}c_{73}c_{74}c_{75}c_{76}+c_{62}c_{73}c_{74}c_{75}c_{76}\\ &\quad=c_{72}c_{73}c_{76}+c_{72}c_{73}c_{75}-c_{72}c_{73}c_{75}c_{76}+c_{72}c_{73}c_{74}\\ &\qquad-c_{72}c_{73}c_{74}c_{76}-c_{72}c_{73}c_{74}c_{75}.\end{aligned}\tag{5.17}$$

Moreover, for $k=4$ and $g=1$,

$$-\sum_{i_5}^{7}c_{i_1 2}c_{75}c_{76}=c_{72}c_{76}+c_{72}c_{75}-c_{72}c_{75}c_{76},\tag{5.18}$$

for $k=4$ and $g=2$,

$$\begin{aligned}&\sum_{i_1=5}^{6}\sum_{i_2=i_1+1}^{7}\sum_{j_2=3}^{4}c_{i_2 j_2}c_{i_1 2}c_{75}c_{76}\\ &\quad=\sum_{j_2=3}^{4}c_{6j_2}c_{52}c_{75}c_{76}+\sum_{j_2=3}^{4}c_{7j_2}c_{52}c_{75}c_{76}+\sum_{j_2=3}^{4}c_{7j_2}c_{62}c_{75}c_{76}\\ &\quad=\sum_{j_2=3}^{4}c_{7j_2}c_{72}-\sum_{j_2=3}^{4}c_{7j_2}c_{72}c_{76}-\sum_{j_2=3}^{4}c_{7j_2}c_{72}c_{75},\end{aligned}\tag{5.19}$$

and when $k=4$ and $g=3$,

$$-c_{52}c_{63}c_{74}c_{75}c_{76}=-c_{72}c_{73}c_{74}.\tag{5.20}$$

For $k=5$ and $g=1$,

$$-\sum_{i_1=6}^{7}c_{i_1 2}c_{76}=c_{72}-c_{72}c_{76},\tag{5.21}$$

and finally for $k = 5$ and $g = 2$,

$$\sum_{j_2=3}^{5} c_{7j_2} c_{62} c_{76} = -\sum_{j_2=3}^{5} c_{7j_2} c_{72}. \tag{5.22}$$

By summing (5.16) - (5.22) we obtain

$$\begin{aligned} &\sum_{k=3}^{5} (R^{-1} Z^U)_{1k} (Z^V R)_{k6} \\ &= c_{72} - c_{72}c_{74}c_{75}c_{76} - c_{72}c_{73}c_{75}c_{76} - c_{72}c_{73}c_{74}c_{75} - c_{72}c_{73}c_{74}c_{76}. \end{aligned} \tag{5.23}$$

Now we return to (5.14) when $n = 7$ and calculate

$$(Z^V R)_{16} = c_{72}c_{73}c_{74}c_{75}c_{76}, \tag{5.24}$$

$$\begin{aligned} (Z^V R)_{12}(Z^V R)_{26} &= \sum_{i_1=3}^{7} c_{i_1 2} c_{73}c_{74}c_{75}c_{76} \\ = c_{72}c_{73}c_{74}c_{75}c_{76} - c_{72}c_{74}c_{75}c_{76} &- c_{72}c_{73}c_{75}c_{76} - c_{72}c_{73}c_{74}c_{76} - c_{72}c_{73}c_{74}c_{75}, \end{aligned} \tag{5.25}$$

and

$$(R^{-1} Z^U)_{16} = -c_{72}. \tag{5.26}$$

Thus, by using (5.23) - (5.26) we see that (5.14) is true when $n = 7$. The general case follows by copying the above approach.

In the next we will verify that $T_n = U_T \Lambda V_T$ which is the same as showing

$$(I + Z^V R) R^{-1} T R = \Lambda (I + Z^V R). \tag{5.27}$$

Direct calculations via Corollary 4.1 give

$$(R^{-1} T R)_{il} = (n - i) I_{\{i=l\}} + \sum_{m=l+1}^{n} c_{mi+1} \prod_{s=i+2}^{l} b_{ms}^{-1} I_{\{i<l\}} \tag{5.28}$$

and from (5.12) it follows that

$$(I + Z^V R)_{ki} = I_{\{k=i\}} + \sum_{i_1 \leq \cdots \leq i_{i-k}}^{[i+1,n]} \prod_{r=1}^{i-k} c_{i_r k+r} I_{\{k<i, k\leq n-2, i\leq n-1\}}. \tag{5.29}$$

Thus (5.27) is true if

$$\sum_{i=1}^{n-1}(I+Z^VR)_{ki}(R^{-1}TR)_{il} = (n-k)(I+Z^VR)_{kl}$$

which by (5.28) and (5.29) can be written

$$\begin{aligned}\sum_{i=k}^{l}\{I_{\{k=i\}} + \sum_{i_1=i+1}^{n}\cdots\sum_{i_{i-k}=i_{i-k-1}}^{n}\prod_{r=1}^{i-k}c^{-1}_{i_rk+r}I_{\{k<i,k\le n-2,i\le n-1\}}\}\\ \times\{(n-i)I_{\{i=l\}} + \sum_{m=l+1}^{n}c_{mi+1}\prod_{s=i+2}^{l}b^{-1}_{ms}I_{\{i<l\}}\}\\ = (n-k)(I+Z^VR)_{kl}.\end{aligned} \tag{5.30}$$

Note that (5.30) immediately holds if $k=l$. For the off-diagonal elements in (5.30) one has to perform rather lengthy calculations and frequently use Theorem 2.2 (ii) and (iii). We indicate a proof based on induction and therefore define the following four $n\times n$ matrices, $G=(g_{kl})$, $H=(h_{kl})$, $(I+Z^VR)^{n-1} = ((I+Z^VR)^{n-1}_{kl})$ and $(R^{-1}TR)^{n-1} = ((R^{-1}TR)^{n-1}_{kl})$;

$$\begin{aligned} g_{kl} &= \sum_{i_1\le\cdots\le i_{l-k-1}}^{[l+1,n]}\prod_{r=1}^{l-k-1}c_{i_rk+r}c_{nl},\\ h_{kl} &= I_{\{k=l\}} + c_{nk+1}\prod_{s=k+2}^{l}b^{-1}_{ns},\\ (I+Z^VR)^{n-1}_{kl} &= I_{\{k=l\}} + \sum_{i_1\le\cdots\le i_{l-k-1}}^{[l+1,n-1]}\prod_{r=1}^{l-k}c_{i_rk+r},\\ (R^{-1}TR)^{n-1}_{kl} &= (n-k-1)I_{\{k=l\}} + \sum_{m=l+1}^{n-1}c_{mk+1}\prod_{s=k+2}^{l}b^{-1}_{ms}I_{\{k<l\}}.\end{aligned}$$

In the above relations we assume $k\le n-1$ and $l\le n-1$, and additionally it will be assumed that if $k=n$ or $l=n$ then the above four mentioned matrices will all equal 0.

Assuming that (5.30) holds for $n-1$ we obtain

$$\sum_{i=k}^{l}(I+Z^V R)_{ki}(R^{-1}TR)_{il}$$
$$= \sum_{i=k}^{l}(I+Z^V R)_{ki}^{n-1}(R^{-1}TR)_{il}^{n-1} + \sum_{i=k}^{l}(I+Z^V R)_{ki}^{n-1}h_{il}$$
$$+ \sum_{i=k}^{l} g_{ki}(R^{-1}TR)_{il}^{n-1} + \sum_{i=k}^{l} g_{ki}h_{il}$$
$$= (n-k)(I+Z^V R)_{kl}^{n-1} + (n-k)g_{kl} + (k-l)g_{kl} + \sum_{i=k+1}^{l}(I+Z^V R)_{ki}^{n-1}h_{il}$$
$$+ \sum_{i=k+1}^{l-1} g_{ki}(R^{-1}TR)_{il}^{n-1} + \sum_{i=k+1}^{l-1} g_{ki}h_{il} + h_{kl}. \tag{5.31}$$

Since

$$(n-k)\{(I+Z^V R)_{kl}^{n-1} + g_{kl}\} = (n-k)(I+Z^V R)_{kl}$$

it has to be shown that

$$(l-k)g_{kl} = \sum_{i=k}^{l}(I+Z^V R)_{ki}^{n-1}h_{il} + \sum_{i=k}^{l} g_{ki}(R^{-1}TR)_{il}^{n-1} + \sum_{i=k}^{l} g_{ki}h_{il} + h_{kl}, \tag{5.32}$$

which is fairly straightforward. The main idea is to use Theorem 2.2 (ii) and (iii). We omit these calculations for the general case and only focus on $n = 5$. In particular we will show that (5.32) is true for element (k,l) when $k = 1$ and $l = 2,3,4$.

For element $(1,2)$ equation (5.32) is trivially true. When considering element $(1,3)$ equation (5.32) reduces to

$$2\sum_{i=4}^{5} c_{i_1}2c_{53} = \sum_{i=3}^{4} c_{i_1}2c_{53} + c_{52}c_{43} + c_{52}c_{53} + c_{52}b_{53}^{-1}. \tag{5.33}$$

Now, applying Theorem 2.2 (ii) and (iii) the right hand side of (5.33) equals

$$-c_{52} + c_{42}c_{53} + c_{52}c_{43} + c_{52}c_{53} + c_{52}b_{53}^{-1} = 2c_{42}c_{53} + 2c_{52}c_{53}$$

and thus (5.33) holds. Continuing with element $(1,4)$ in (5.32) we get that (5.32) is equivalent to

$$\begin{aligned} 3c_{52}c_{53}c_{54} &= \sum_{i=3}^{4} c_{i_1 2}c_{53}b_{54}^{-1} + c_{42}c_{43}c_{54} + c_{52}c_{53}b_{54}^{-1} \\ &\quad + \sum_{i=4}^{5} c_{i_1 2}c_{53}c_{54} + c_{52}b_{53}^{-1}b_{54}^{-1}. \end{aligned} \tag{5.34}$$

Utilizing Theorem 2.2 (ii) and (iii) implies that the following calculations for the right hand side of (5.34) can be carried through;

$$\begin{aligned} &-c_{52}b_{54}^{-1} + c_{42}c_{53}b_{54}^{-1} - c_{52}c_{43} + c_{52}c_{53}b_{54}^{-1} - c_{52}c_{53} + c_{52}c_{53}c_{54} \\ &= c_{52}c_{53}b_{54}^{-1} + c_{42}c_{53}c_{54} + 2c_{52}c_{53}c_{54} \\ &= c_{52}c_{53}b_{54}^{-1} - c_{52}c_{53} + 2c_{52}c_{53}c_{54} = 3c_{52}c_{53}c_{54}. \end{aligned}$$

Similarly we may show that (5.32) is true for $k=2, l=3,4$ and $k=3, l=4$. □

Now we are ready to present eigenvectors of the matrix B.

Theorem 5.2 *Let $B \in \mathbb{B}_n$ and the matrices U_n, V_n and T_n be given by* (4.1), (4.2) *and Theorem* 4.4, *respectively. Then the matrix of left eigenvectors $W_L = (w_{ij}^L)$ for B is given by*

$$w_{i1}^L = \prod_{k=2}^{i} b_{1k} \prod_{l=i+1}^{n} b_{l1}, \; i = 1, \ldots, n, \tag{5.35}$$

$$w_{ii}^L = (-b_{1i})^{I_{\{i>1\}}} \prod_{k=2}^{i-1} b_{ik} \prod_{l=i+1}^{n} b_{li}, \tag{5.36}$$

$$w_{ij}^L = (-1)^{I_{\{i>j\}}} b_{1j} \prod_{\substack{k=2 \\ k\neq j}}^{i} b_{jk} \prod_{\substack{l=i+1 \\ l\neq j}}^{n} b_{lj}, \qquad j > 1. \tag{5.37}$$

Proof: Let W_L^n be W_L where n indicates the size of the matrix. Thus, from (5.35)–(5.37), it follows that

$$\begin{aligned} W_L^n &= \sum_{i=1}^{n} \prod_{k=2}^{i} b_{1k} \prod_{l=i+1}^{n} b_{l1} e_i e_1' + \sum_{i=1}^{n} (-b_{1i})^{I_{\{i>1\}}} \prod_{k=2}^{i-1} b_{ik} \prod_{l=i+1}^{n} b_{li} e_i e_i' \\ &\quad - \sum_{i=3}^{n} \sum_{j=2}^{i-1} b_{1j} \prod_{\substack{k=2 \\ k\neq j}}^{i} b_{jk} \prod_{\substack{l=i+1 \\ l\neq j}}^{n} b_{lj} e_i e_j' + \sum_{j=2}^{n} \sum_{i=1}^{j} b_{1j} \prod_{\substack{k=2 \\ k\neq j}}^{i} b_{jk} \prod_{\substack{l=i+1 \\ l\neq j}}^{n} b_{lj} e_i e_j'. \end{aligned} \tag{5.38}$$

From Theorem 4.1 it follows that

$$B = V_n T_n U_n,$$

and from Theorem 5.1

$$T_n = U_T \Lambda V_T.$$

Hence, $V_T U_n$ satisfies

$$V_T U_n B = \Lambda V_T U_n,$$

which means that $V_T U_n$ is a matrix consisting of left eigenvectors. We are going to show that

$$W_L^n = V_T U_n.$$

From (5.38) it follows that

$$W_L^n = \begin{pmatrix} W_L^{n-1} D_n & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & H_{\bullet n} \\ H_{n\bullet} & h_{nn} \end{pmatrix}, \tag{5.39}$$

where

$$\begin{aligned} D_n &= \sum_{i=1}^{n-1} b_{ni} d_i d_i', \ d_i : (n-1) \times 1, \\ H_{\bullet n} &= (h_{in}) = \sum_{i=1}^{n-1} b_{1n} \prod_{k=2}^{i} b_{nk} \prod_{l=i+1}^{n-1} b_{ln} d_i, \\ H_{n\bullet} &= (h_{ni}) = \prod_{k=2}^{n} b_{1k} d_1' - \sum_{j=2}^{n-1} b_{1j} \prod_{\substack{k=2 \\ k \neq j}}^{n} b_{jk} d_j, \\ h_{nn} &= -b_{1n} \prod_{k=2}^{n-1} b_{nk}. \end{aligned} \tag{5.40}$$

For small n, say $n \leq 6$, it is fairly easy to show that $W_L^n = V_T U_n$. Because of (5.39), we will indicate a proof for a general n, which is based on induction. Suppose that

$$W_L^{n-1} = (R_{n-1}^{-1} + Z_{n-1}^V) U_{n-1},$$

where it has been used that $V_T = R_{n-1}^{-1} + Z_{n-1}^V$ (see 5.5), and it will be shown that

$$W_L^n = (R_n^{-1} + Z_n^V)U_n,$$

which for small n directly can be verified. First observe that

$$(H_{n\bullet} : h_{nn}) = e_n' U_n = W_L^{n\bullet},$$

where $e_n : n \times 1$ and $W_L^{n\bullet}$ is the last row in W_L^n. Let $W_L^n = (w_{jj}^n)$ and $U_{\bullet n}$ stands for the last column in U_n. Then

$$w_{n-1n}^n = (0, 0, \ldots, 0, -1)U_{\bullet n} = h_{n-1n}. \tag{5.41}$$

It remains to show that for the first $n-2$ elements of $H_{\bullet n}$

$$h_{in} = ((R_n^{-1} + Z_n^V)U_n)_{in}, \quad i = 1, \ldots, n-2, \tag{5.42}$$

and

$$((R_{n-1}^{-1} + Z_{n-1}^V)U_{n-1}D_{n-1})_{ij} = ((R_n^{-1} + Z_n^V)U_n)_{ij}, \quad i, j = 1, \ldots, n-1. \tag{5.43}$$

First consider (5.43). As in the proof of Theorem 5.1 we will use the products $Z_n^V R_n$ and $R_n^{-1}U_n$, and thus work with

$$(I + Z_n^V R_n)R_n^{-1}U_n, \tag{5.44}$$

which is an algebraic relation simpler than $(R_n^{-1} + Z_n^V)U_n$. The expression for $Z_n^V R_n$ was given in (5.12), and thus

$$I + Z_n^V R_n = I + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{i_1 \leq \cdots \leq i_{j-1}}^{[j+1,n]} \prod_{m=1}^{j-i} b_{i_m i+m}^{-1} b_{i+m i_m} e_i e_j'. \tag{5.45}$$

Moreover, (4.1) gives that

$$\begin{aligned} R_n^{-1}U_n &= \sum_{i=1}^{n} \prod_{m=2}^{i} b_{1m} b_{i+11}^{I_{\{i<n\}}} e_i e_1' + \sum_{j=2}^{n} b_{1j} \prod_{m=2}^{j-1} b_{jm} e_{j-1} e_j' \\ &\quad - \sum_{j=2}^{n} \sum_{i=j}^{n} b_{1j} \prod_{\substack{m=2 \\ m \neq j}}^{i} b_{jm} b_{i+1j}^{I_{\{i<n\}}} e_i e_j'. \end{aligned} \tag{5.46}$$

We indicate now how to prove (5.43). However, a few details will just be presented. For example, it will be shown that the first column of the left hand side of (5.43) equals the first column of the right hand side. It is, besides lengthy calculations, fairly easy to verify (5.43) for the other columns too.

From (5.45) and (5.46) it follows that the first column in (5.43) can be written

$$\{\sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2}\sum_{i_1\le\cdots\le i_{j-i}}^{[j+1,n-1]}\prod_{m=1}^{j-i} b^{-1}_{i_m i+m} b_{i+m i_m} e_i e_j' + e_i e_i'\}\{\sum_{k=1}^{n-1}\prod_{\underline{m}=2}^{k} b_{1\underline{m}} b_{k+11} b_{n1} e_k\}$$

$$(5.47)\quad = \{\sum_{i=1}^{n-3}\sum_{j=i+1}^{n-1}\sum_{i_1\le\cdots\le i_{j-i}}^{[j+1,n]}\prod_{m=1}^{j-i} b^{-1}_{i_m i+m} b_{i+m i_m} e_i e_j' + e_1 e_1'\}\{\sum_{k=1}^{n}\prod_{\underline{m}=2}^{k} b_{1\underline{m}} b_{k+11} e_k\}.$$

Furthermore, (5.47) is equivalent to

$$\prod_{\underline{m}=2}^{i} b_{1\underline{m}} b_{i+11} b_{n1} + \sum_{j=i+1}^{n-2}\sum_{i_1\le\cdots\le i_{j-i}}^{[j+1,n-1]}\prod_{m=1}^{j-i} b^{-1}_{i_m m+i} b_{m+i i_m}\prod_{\underline{m}=2}^{j} b_{1\underline{m}} b_{j+11} b_{n1}$$

$$(5.48)\quad = \prod_{\underline{m}=2}^{i} b_{1\underline{m}} b_{i+11} + \sum_{j=i+1}^{n-1}\sum_{i_1\le\cdots\le i_{j-i}}^{[j+1,n]}\prod_{m=1}^{j-i} b^{-1}_{i_m m+i} b_{m+i i_m}\prod_{\underline{m}=2}^{j} b_{1\underline{m}} b_{j+11}.$$

The validity of (5.48) will be exploited and first the left hand side is considered. Some manipulations give that it equals

$$\prod_{\underline{m}=2}^{i} b_{1\underline{m}}(b_{i+11} + \sum_{j=i+1}^{n-2}\sum_{i_1\le\cdots\le i_{j-i}}^{[j+1,n-1]}\prod_{m=i+1}^{j} b^{-1}_{i_{m-i} m} b_{m i_{m-i}} b_{1m} b_{j+11}) b_{n1}$$

which by applying Theorem 2.3 can be written

$$\prod_{\underline{m}=2}^{i} b_{1\underline{m}}\prod_{i+1}^{n-1} b_{m1} b_{n1} = \prod_{\underline{m}=2}^{i} b_{1\underline{m}}\prod_{i+1}^{n} b_{m1}.$$

When performing the same calculations to the right hand side of (5.48) we see that (5.48) is true.

Finally, (5.42) is briefly considered. From (5.45) and (5.46) it follows that

$$\{(R_n^{-1} + Z_n^V)U_n\}_{in} = \{(I + Z_n^V R_n)R_n^{-1}U_n\}_{in} = \{Z_n^V R_n R_n^{-1} U_n\}_{in}$$
$$= \prod_{m=1}^{n-1-i} b^{-1}_{ni+m} b_{i+mn} b_{1n}\prod_{\underline{m}=2}^{n-1} b_{n\underline{m}} = b_{1n}\prod_{m=i+1}^{n-1} b^{-1}_{nm} b_{mn} b_{1n}\prod_{\underline{m}=i+1}^{n-1} b_{n\underline{m}}\prod_{\underline{m}=2}^{i} b_{n\underline{m}}$$
$$= b_{1n}\prod_{m=i+1}^{n-1} b_{mn}\prod_{\underline{m}=2}^{i} b_{n\underline{m}},$$

which implies that (5.40) as well as (5.42) hold. □

Theorem 5.3 *Let* $B \in \mathbb{B}_n$ *and* $c_{ij} = b_{ji}^{-1} b_{ij}$, $i \neq j$. *Then the matrix of left eigenvectors,* W_L, *defined in Theorem 5.2 can be decomposed as* $W_L = D_L V D_R$, *where* D_L *and* D_R *are diagonal matrices, and* V *is a Vandermonde matrix given by*

$$(5.49)\quad D_L = \sum_{i=1}^{n} \prod_{k=2}^{i} c_{k1} e_i e_i'$$

$$(5.50)\quad D_R = \prod_{l=2}^{n} b_{l1} e_1 e_1' + \sum_{i=2}^{n} b_{1i} \prod_{\substack{l=2\\ l\neq i}}^{n} b_{li} e_i e_i' = \sum_{i=1}^{n} b_{1i}^{I_{\{i>1\}}} \prod_{\substack{l=2\\ l\neq i}}^{n} b_{li} e_i e_i'$$

$$(5.51)\quad V = \sum_{i=1}^{n} e_i e_1' + \sum_{i=1}^{n} \sum_{j=2}^{n} (-1)^{i-1} c_{1j}^{i-1} e_i e_j' = \sum_{i=1}^{n} \sum_{j=1}^{n} ((-c_{1j})^{i-1})^{I_{\{j>1\}}} e_i e_j'.$$

Proof: The matrix W_L, defined in Theorem 5.2, can be written

$$W_L = D_L D_L^{-1} \{ \sum_{i=1}^{n} \prod_{k=2}^{i} c_{k1} e_i e_1' - \sum_{i=2}^{n} \prod_{k=2}^{i-1} c_{ki} e_i e_i' + \sum_{i=1}^{n} (-1)^{I_{\{i>j\}}} \prod_{\substack{k=2\\ k\neq j}}^{n} c_{kj} e_i e_j' \} D_R.$$

Thus, it has to be shown that

$$D_L^{-1} \{ \sum_{i=1}^{n} \prod_{k=2}^{i} c_{k1} e_i e_1' - \sum_{i=2}^{n} \prod_{k=2}^{i-1} c_{ki} e_i e_i' + \sum_{i=1}^{n} (-1)^{I_{\{i>j\}}} \prod_{\substack{k=2\\ k\neq j}}^{n} c_{kj} e_i e_j' \} = V,$$

where

$$D_L^{-1} = \sum_{i=1}^{n} \prod_{k=2}^{i} c_{1k} e_i e_i'.$$

However, from Theorem 2.2 (ii) it follows that

$$\prod_{k=2}^{i} c_{1k} \prod_{l=2}^{i-1} c_{li} = (-1)^{i-2} c_{1i}^{i-1}$$

which verifies the theorem. □

When obtaining the right eigenvectors for $B \in \mathbb{B}_n$ Theorem 5.3 will be utilized. Therefore a general expression for the inverse of a Vandermonde matrix is required, which is well-known (e.g. see El-Mikkaway, 2003). Indeed the technique applied by Bondesson & Traat (2007) when finding their results is similar to the one which usually is applied when finding for example the inverse of the Vandermonde matrix. In the next lemma we present, without a proof, the result for the inverse Vandermonde matrix.

Lemma 5.2 *Let V be given in Theorem 5.3. Then $W = (w_{ij})$, where $W = V^{-1}$, is given by*

$$\begin{aligned} w_{ij} &= \sum_{\substack{i_1<\cdots<i_{n-j} \\ i_1\neq i,\ldots,i_{n-j}\neq i}}^{[1,n]} (-c_{1i_1})^{I_{\{i_1>1\}}} \prod_{m=2}^{n-j} (-c_{1i_m})(-1)^{n-j} \\ &\qquad \times \prod_{\substack{k=2 \\ k\neq i}}^{n} ((c_{1k} + (-c_{1i})^{I_{\{i>1\}}})(-c_{1i}-1)^{I_{\{i>1\}}})^{-1}, \qquad j \leq n-1 \\ w_{in} &= \prod_{\substack{k=2 \\ k\neq i}}^{n} ((c_{1k} + (-c_{1i})^{I_{\{i>1\}}})(-c_{1i}-1)^{I_{\{i>1\}}})^{-1}. \end{aligned}$$

Remark 5.1 *If $Q = (q_{ij})$ is a Vandermomde matrix, where $q_{ij} = (c_j)^{I_{\{i>1\}}}$, $c_k \neq c_l$, its inverse may immediately be obtained by noting that*

$$\begin{aligned} \sum_{j=1}^{n} \sum_{\substack{i_1<\cdots<i_{n-j} \\ i_1\neq i,\ldots,i_{n-j}\neq i}}^{[1,n]} \prod_{m=1}^{n-j} c_{i_m} c_i^{j-1} (-1)^{n-j} &= \prod_{\substack{j=1 \\ j\neq i}}^{n} (c_i - c_j), \\ \sum_{j=1}^{n} \sum_{\substack{i_1<\cdots<i_{n-j} \\ i_1\neq i,\ldots,i_{n-j}\neq i}}^{[1,n]} \prod_{m=1}^{n-j} c_{i_m} c_l^{j-1} (-1)^{n-j} &= 0, \qquad l \neq i. \end{aligned}$$

Before stating the last theorem it is noted that by applying Theorem 2.2

$$\begin{aligned} &\prod_{\substack{k=2 \\ k\neq i}}^{n} ((c_{1k} + (-c_{1i})^{I_{\{1>i\}}})(-c_{1i}-1)^{I_{\{1>i\}}})^{-1} \\ &= ((-c_{i1})^{n-2}(-b_{1i}))^{I_{\{i>1\}}} \prod_{\substack{k=2 \\ k\neq i}}^{n} b_{ik}. \end{aligned} \tag{5.52}$$

Note that the left hand side of this expression appears in Lemma 5.2. Using (5.52) a theorem concerning the right eigenvectors will be formulated and proved.

Theorem 5.4 *Let $B \in \mathbb{B}_n$. Then the matrix of right eigenvectors $W_R = (w_{ij}^R)$ for B is the following:*

$$
\begin{aligned}
w_{1j}^R &= \prod_{l=j+1}^{n} c_{l1} \sum_{i_1<\cdots<i_{n-j}}^{[2,n]} \prod_{m=1}^{n-j} c_{1i_m}, \qquad j = 1, 2, \ldots, n-1, \\
w_{1n}^R &= 1, \qquad w_{in}^R = -c_{1i}, \qquad i = 2, 3, \ldots, n, \\
w_{ij}^R &= \prod_{\substack{l=j+1 \\ l\neq i}}^{n} c_{l1} c_{1i}^{I_{\{i\leq j\}}} \sum_{\substack{i_1<\cdots<i_{n-j} \\ i_1\neq i,\ldots,i_{n-j}\neq i}}^{[1,n]} (-c_{1i_1})^{I_{\{i_1>1\}}} \prod_{m=2}^{n-j} c_{1i_m}, \\
&\qquad\qquad i = 2, 3, \ldots, n, \quad j = 1, 2, \ldots, n-1.
\end{aligned}
$$

Proof: First the inverse of D_R, given by (5.50), will be multiplied with (5.52) which yields

$$
(-1)^{I_{\{i>1\}}} \sum_{i=1}^{n} \prod_{\substack{l=2 \\ l\neq i}}^{n} c_{l1} e_i e_i'. \tag{5.53}
$$

Moreover, multiplying

$$
\prod_{\substack{k=2 \\ k\neq i}}^{n} (c_{1k} + (-c_{1i})^{I_{\{1>i\}}})(-c_{1i} - 1)^{I_{\{1>i\}}} V,
$$

where V is the Vandermonde matrix presented in Theorem 5.3, with (5.53) from the left and with the inverse of D_L, given by (5.49), from the right we obtain after some manipulations

$$
\begin{aligned}
&\sum_{i=1}^{n} (-c_{1i})^{I_{\{i>1\}}} e_i e_n' + \sum_{j=1}^{n-1} \prod_{l=j+1}^{n} c_{l1} \sum_{i_1<\cdots<i_{n-j}}^{[2,n]} \prod_{m=1}^{n-j} c_{1i_m} e_1 e_j' \\
&+ \sum_{i=2}^{n} \sum_{j=1}^{n-1} \prod_{\substack{l=j+1 \\ l\neq i}}^{n} c_{l1} c_{1i}^{I_{\{i\leq j\}}} \sum_{\substack{i_1<\cdots<i_{n-j} \\ i_1\neq i,\ldots,i_{n-j}\neq i}}^{[1,n]} (-c_{1i_1})^{I_{\{i_1>1\}}} \prod_{m=2}^{n-j} c_{1i_m} e_i e_j'.
\end{aligned}
$$

This relation establishes the statement of the theorem. □

The theorem can be further simplified but this will be omitted because there is no really use of it and it burdens the presentation. However, in the next example one can see how the matrix given in (5.56) reduces to the matrix in (5.55).

Example 5.1. In this example Theorem 2.2 will frequently be utilized. According to Theorem 5.2 the matrix of left eigenvectors, in the case $n = 4$, equals

$$\begin{pmatrix} b_{21}b_{31}b_{41} & b_{32}b_{42}b_{12} & b_{13}b_{23}b_{43} & b_{14}b_{24}b_{34} \\ b_{12}b_{31}b_{41} & -b_{12}b_{32}b_{42} & b_{13}b_{32}b_{43} & b_{14}b_{42}b_{34} \\ b_{12}b_{13}b_{41} & -b_{12}b_{23}b_{42} & -b_{13}b_{32}b_{43} & b_{14}b_{42}b_{43} \\ b_{12}b_{13}b_{14} & -b_{12}b_{23}b_{24} & -b_{13}b_{32}b_{34} & -b_{14}b_{42}b_{43} \end{pmatrix},$$

which can be decomposed as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{21} & 0 & 0 \\ 0 & 0 & c_{21}c_{31} & 0 \\ 0 & 0 & 0 & c_{21}c_{31}c_{41} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -c_{12} & -c_{13} & -c_{14} \\ 1 & c_{12}^2 & c_{13}^2 & c_{14}^2 \\ 1 & -c_{12}^3 & -c_{13}^3 & -c_{14}^3 \end{pmatrix}$$

$$\times \begin{pmatrix} b_{21}b_{31}b_{41} & 0 & 0 & 0 \\ 0 & b_{12}b_{32}b_{42} & 0 & 0 \\ 0 & 0 & b_{13}b_{23}b_{43} & 0 \\ 0 & 0 & 0 & b_{14}b_{24}b_{34} \end{pmatrix}. \tag{5.54}$$

The inverse of the Vandermonde matrix in this expression equals

$$\begin{pmatrix} b_{12}b_{13}b_{14} & 0 & 0 & 0 \\ 0 & -c_{21}^2 b_{12}b_{23}b_{24} & 0 & 0 \\ 0 & 0 & -c_{31}^2 b_{13}b_{32}b_{34} & 0 \\ 0 & 0 & 0 & -c_{21}^2 b_{14}b_{42}b_{43} \end{pmatrix}$$

$$\times \begin{pmatrix} c_{12}c_{13}c_{14} & c_{12}c_{13} + c_{12}c_{14} + c_{13}c_{14} & c_{12} + c_{13} + c_{14} & 1 \\ -c_{13}c_{14} & -c_{13} - c_{14} + c_{13}c_{14} & c_{13} + c_{14} - 1 & 1 \\ -c_{12}c_{14} & -c_{12} - c_{14} + c_{12}c_{14} & c_{12} + c_{14} - 1 & 1 \\ -c_{12}c_{13} & -c_{12} - c_{13} + c_{12}c_{13} & c_{12} + c_{13} - 1 & 1 \end{pmatrix}.$$

Therefore, when taking the inverse of (5.54), the right eigenvectors are given by

$$\begin{pmatrix} c_{21}c_{31}c_{41} & 0 & 0 & 0 \\ 0 & -c_{21}^2c_{32}c_{42} & 0 & 0 \\ 0 & 0 & -c_{31}^2c_{23}c_{43} & 0 \\ 0 & 0 & 0 & -c_{41}^2c_{24}c_{34} \end{pmatrix}$$

$$\times \begin{pmatrix} c_{12}c_{13}c_{14} & c_{12}c_{13}+c_{12}c_{14}+c_{13}c_{14} & c_{12}+c_{13}+c_{14} & 1 \\ -c_{13}c_{14} & -c_{13}-c_{14}+c_{13}c_{14} & c_{13}+c_{14}-1 & 1 \\ -c_{12}c_{14} & -c_{12}-c_{14}+c_{12}c_{14} & c_{12}+c_{14}-1 & 1 \\ -c_{12}c_{13} & -c_{12}-c_{13}+c_{12}c_{13} & c_{12}+c_{13}-1 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{12} & 0 & 0 \\ 0 & 0 & c_{12}c_{13} & 0 \\ 0 & 0 & 0 & c_{12}c_{13}c_{14} \end{pmatrix}$$

$$= \begin{pmatrix} c_{21}c_{31}c_{41} & 0 & 0 & 0 \\ 0 & -c_{31}c_{41} & 0 & 0 \\ 0 & 0 & -c_{21}c_{31} & 0 \\ 0 & 0 & 0 & -c_{21}c_{31} \end{pmatrix}$$

$$\times \begin{pmatrix} c_{12}c_{13}c_{14} & c_{12}^2c_{13}+c_{12}c_{14}+c_{13}c_{14} & c_{12}^2c_{13}+c_{12}c_{13}^2+c_{12}c_{13}c_{14} & c_{12}c_{13}c_{14} \\ -c_{13}c_{14} & -c_{13}c_{12}-c_{14}c_{12}+c_{12}c_{13}c_{14} & c_{12}c_{13}^2+c_{12}c_{13}c_{14}-c_{12}c_{13} & c_{12}c_{13}c_{14} \\ -c_{12}c_{14} & -c_{12}^2-c_{12}c_{14}+c_{12}^2c_{14} & c_{12}^2c_{13}+c_{12}c_{13}c_{14}-c_{12}c_{13} & c_{12}c_{13}c_{14} \\ -c_{12}c_{13} & -c_{12}^2-c_{12}c_{13}+c_{12}^2c_{13} & c_{12}^2c_{13}+c_{12}c_{13}^2-c_{12}c_{13} & c_{12}c_{13}c_{14} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -c_{42}-c_{32}+1 & 1-c_{43}-c_{42} & 1 \\ 1 & -c_{42}-c_{32}-c_{12} & c_{12}c_{43}-c_{12}-c_{42} & -c_{12} \\ 1 & -c_{42}+1-c_{12} & c_{12}c_{43}-c_{13}-c_{43} & -c_{13} \\ 1 & -c_{32}+1-c_{12} & -c_{12}-c_{13}+1 & -c_{14} \end{pmatrix}. \tag{5.55}$$

The last relation will now be compared to what is obtained from Theorem 5.4. It follows from the theorem that

$$W_R = \begin{pmatrix} 1 & c_{31}c_{41}(c_{12}c_{13}+c_{12}c_{14}+c_{13}c_{14}) & c_{41}(c_{12}+c_{13}+c_{14}) & 1 \\ 1 & c_{31}c_{41}c_{12}(c_{13}+c_{14}-c_{13}c_{14}) & c_{41}c_{12}(1-c_{13}-c_{14}) & -c_{12} \\ 1 & c_{41}(c_{12}+c_{14}-c_{12}c_{14}) & c_{41}c_{13}(-c_{12}-c_{14}+1) & -c_{13} \\ 1 & c_{31}(c_{12}+c_{13}-c_{12}c_{13}) & 1-c_{12}-c_{13} & -c_{14} \end{pmatrix} \tag{5.56}$$

which by some manipulations can be shown to be identical to (5.55). For example, by using Theorem 2.2 the element $w_{12}^R = c_{31}c_{41}(c_{12}c_{13}+c_{12}c_{14}+c_{13}c_{14}) = -c_{42} - c_{32} + 1$.

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