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Research Report
Centre of Biostochastics

Swedish University of Agricultural Sciences

Report 2008:4
ISSN 1651-8543

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Abstract

A pair of square matrices A and B of the same size are said to satisfy an inequality $A \geq B$ in Löwner partial ordering if $A - B$ is nonnegative definite. Through generalized inverses of matrices, we give in this paper general expressions of Xs that satisfy the following matrix inequalities $AX + (AX)^* \geq B$, $AXA^* \geq B$, $AXB \geq C$ and $AXB + (AXB)^* \geq C$. Various special cases of the four inequalities are also considered.

Keywords: Matrix equation; matrix inequality in Löwner partial ordering; general solution; generalized inverses of matrices; rank formulas

AMS classification: 15A09; 15A24; 15A39

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1 Introduction

Throughout this paper, $\mathbb{C}$ denotes the field of complex numbers; the symbols A^*, $r(A)$ and $\mathscr{R}(A)$ stand for the conjugate transpose, the rank and the range (column space) of a matrix $A \in \mathbb{C}^{m\times n}$, respectively; $[\,A,\, B\,]$ denotes a row block matrix consisting of A and B. For a pair of square matrices of the same size, the expression $A \geq B (A > B)$ means that $A - B$ is a nonnegative definite (positive definite) matrix. In such a case, the pair of matrices are said to satisfy an inequality in Löwner partial ordering. A nonnegative definite matrix A of order m is said to a contraction if all its eigenvalues are less then or equal to 1, i.e., $0 \leq A \leq I_m$, to be a strict contraction if all its eigenvalues are less then 1, i.e., $0 \leq A < I_m$.

The Moore-Penrose inverse $A^{\dagger}$ of an $m \times n$ matrix A is defined to be the unique solution X to the four Penrose equations

$$\text{(i) } AXA = A, \quad \text{(ii) } XAX = X, \quad \text{(iii) } (AX)^* = AX, \quad \text{(iv) } (XA)^* = XA.$$

For convenience, the symbols E_A and F_A stand for the two orthogonal projectors $E_A = I_m - AA^{\dagger}$ and $F_A = I_n - A^{\dagger}A$.

The Löwner partial ordering is one of the most basic concepts in matrix theory. This concept is widely used to characterize relation between Hermitian (symmetric) matrices. Numerous results on Löwner partial ordering and its applications can be found in the literature. Motivated by the work on solving inequalities in elementary mathematics, it is reasonable to propose a general research topic on solving matrix inequalities in Löwner partial ordering. For instance, suppose $f(X)$ is a square matrix function, where X is a variable matrix, and suppose M is a Hermitian matrix. Then the problem is to find Xs that satisfy the following inequalities

$$f(X) \geq M \quad \text{or} \quad f(X) \leq M.$$

In fact, finding solutions with definiteness to matrix equations can be regarded a special cases of solving matrix equalities in Löwner partial ordering. As an introductory work, we study in this paper the following four simple matrix inequalities

(I) $AX + (AX)^* \geq B$ for $B = B^*$.

(II) $AXA^* \geq B$ for $B = B^*$.

(III) $AXB \geq C$ for $C = C^*$.

(IV) $AXB + (AXB)^* \geq C$ for $C = C^*$.

Speaking clearly, we shall consider the following problems on the four inequalities:

(a) Necessary and sufficient conditions for the inequalities to be consistent.

(b) General expressions of X that satisfies the inequalities when the inequalities are consistent.

(c) General solutions to the inequalities when B and C on the right-hands are nonnegative definite, negative definite, and null.

(d) Properties of solutions to the inequalities.

Most of the results obtained in the paper are new and valuable for demonstrating relations of matrices in Löwner partial ordering.

2 Preliminaries

In this section, we give some known results on Löwner partial ordering, general solutions to matrix equations, and rank formulas for partitioned matrices and linear matrix pencils. We shall use these results in Sections 3–6 for solving the inequalities in Section 1.

Lemma 2.1 *If both* $A \geq B \geq 0$, *then* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$.

Lemma 2.2 *Let* $A \in \mathbb{C}^{m\times m}$ *and* $B \in \mathbb{C}^{m\times n}$.

(a) *If* $A \geq 0$, *then* $A^{\dagger} \geq 0$ *and* $B^*AB \geq 0$.

(b) *If* $I_m - A \geq 0$, *then* $I_m - BB^{\dagger}ABB^{\dagger} \geq 0$.

(c) *If* $I_m - A > 0$, *then* $I_m - BB^{\dagger}ABB^{\dagger} > 0$.

Lemma 2.3 *Let* $0 \leq A \in \mathbb{C}^{m\times m}, 0 \leq C \in \mathbb{C}^{p\times p}$ *and* $B \in \mathbb{C}^{m\times p}$. *Then,*

(a) $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0 \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $C \geq B^*A^{\dagger}B \Leftrightarrow \mathscr{R}(B^*) \subseteq \mathscr{R}(C)$ *and* $A \geq BC^{\dagger}B^*$.

(b) $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} > 0 \Leftrightarrow r(A) = m$ *and* $C > B^*A^{-1}B \Leftrightarrow r(C) = p$ *and* $A > BC^{-1}B^*$.

Lemma 2.4 (Tian and Liu, 2006) *Let $A \in \mathbb{C}^{m\times n}$ and $B = B^* \in \mathbb{C}^{m\times m}$ be given. Then,*

(a) *The matrix equation*
$$AX + (AX)^* = B \tag{2.1}$$
has a solution if and only if $E_A B E_A = 0$. In this case, the general solution to the equation is
$$X = \frac{1}{2}A^{\dagger}B(\, 2I_m - AA^{\dagger}\,) + VA^* + F_A W, \tag{2.2}$$
where both $V = -V^* \in \mathbb{C}^{n\times n}$ and $W \in \mathbb{C}^{n\times m}$ are arbitrary.

(b) *The matrix equation*
$$AX + (AX)^* = BB^* \tag{2.3}$$
has a solution if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$. In this case, the general solution to the equation is
$$X = \frac{1}{2}A^{\dagger}BB^* + VA^* + F_A W, \tag{2.4}$$
where both $V = -V^* \in \mathbb{C}^{n\times n}$ and $W \in \mathbb{C}^{n\times m}$ are arbitrary.

Lemma 2.5 (Khatri and Mitra, 1976) *Let $A, B \in \mathbb{C}^{m\times n}$ be given. Then,*

(a) *The matrix equation*
$$AX = B \tag{2.5}$$
has a Hermitian solution if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ and $AB^* = BA^*$. In this case, the general Hermitian solution to (2.5) is
$$X = A^{\dagger}B + (A^{\dagger}B)^* - A^{\dagger}BA^{\dagger}A + F_A W F_A, \tag{2.6}$$
where $W = W^* \in \mathbb{C}^{n\times n}$ is arbitrary.

(b) *The matrix equation*
$$AXX^* = B \tag{2.7}$$
has a solution if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A), BA^* \geq 0$, and $r(BA^*) = r(B)$. In this case, the general solution to (2.7) is
$$XX^* = B^*(BA^*)^{\dagger}B + F_A WW^* F_A, \tag{2.8}$$
where $W \in \mathbb{C}^{n\times n}$ is arbitrary.

Lemma 2.6 (Baksalary 1984, Gross 2000, Khatri and Mitra 1976) *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{m\times m}$ *be given. Then,*

(a) *The matrix equation*

$$AXA^* = B \tag{2.9}$$

has a solution if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $\mathscr{R}(B^*) \subseteq \mathscr{R}(A)$, *or equivalently* $E_AB = BE_A = 0$. *In this case, the general solution to* (2.9) *is*

$$X = A^\dagger B(A^\dagger)^* + W - A^\dagger AWA^\dagger A, \tag{2.10}$$

where $W \in \mathbb{C}^{n\times n}$ *is arbitrary.*

(b) *The matrix equation*

$$AXX^*A^* = B \geq 0 \tag{2.11}$$

has a solution if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$. *In this case, the general solution to* (2.11) *is*

$$XX^* = (\, A^\dagger B^{1/2} + F_AW \,)(\, A^\dagger B^{1/2} + F_AW \,)^*, \tag{2.12}$$

where $W \in \mathbb{C}^{n\times n}$ *is arbitrary.*

Lemma 2.7 (Penrose, 1955) *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$ *and* $C \in \mathbb{C}^{m\times q}$ *be given. Then the matrix equation*

$$AXB = C \tag{2.13}$$

has a solution if and only if $\mathscr{R}(C) \subseteq \mathscr{R}(A)$ *and* $\mathscr{R}(C^*) \subseteq \mathscr{R}(B^*)$, *or equivalently,* $E_AC = 0$ *and* $CF_B = 0$. *In this case, the general solution to* (2.13) *is*

$$X = A^\dagger CB^\dagger + W - A^\dagger AWBB^\dagger, \tag{2.14}$$

where $W \in \mathbb{C}^{n\times p}$ *is arbitrary.*

Lemma 2.8 (Marsaglia and Styan, 1974) *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ *and* $D \in \mathbb{C}^{l\times k}$. *Then*

$$r[\, A,\, B\,] = r(A) + r(E_AB) = r(B) + r(E_BA), \tag{2.15}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{2.16}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_BAF_C), \tag{2.17}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r\begin{bmatrix} 0 & E_AB \\ CF_A & D - CA^\dagger B \end{bmatrix}. \tag{2.18}$$

Lemma 2.9 (Tian 2002, Tian and Cheng 2003) *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, *and* $C \in \mathbb{C}^{l\times n}$ *be given. Then the maximal and minimal ranks of* $A - BXC$ *are given by*

$$\max_{X\in\mathbb{C}^{k\times l}} r(\, A - BXC\,) = \min\left\{ r[\, A,\, B\,], \quad r\begin{bmatrix} A \\ C \end{bmatrix} \right\}, \tag{2.19}$$

$$\min_{X\in\mathbb{C}^{k\times l}} r(\, A - BXC\,) = r[\, A,\, B\,] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{2.20}$$

Lemma 2.10 *Let* $0 \leq A \in \mathbb{C}^{m\times m}$ *and* $B \in \mathbb{C}^{m\times k}$ *be given. Then*

$$\max_{0\leq X\in\mathbb{C}^{k\times k}} r(\, A - BXB^*\,) = r[\, A,\, B\,], \tag{2.21}$$

$$\min_{0\leq X\in\mathbb{C}^{k\times k}} r(\, A - BXB^*\,) = r[\, A,\, B\,] - r(B), \tag{2.22}$$

$$\max_{0\leq X\in\mathbb{C}^{k\times k}} r(\, A + BXB^*\,) = r[\, A,\, B\,], \tag{2.23}$$

$$\min_{0\leq X\in\mathbb{C}^{k\times k}} r(\, A + BXB^*\,) = r(A). \tag{2.24}$$

Proof. Let

$$M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ I_n \end{bmatrix}, \quad S_1 = S - MM^{\dagger}S. \tag{2.25}$$

It was shown in Tian and Liu (2006) that $A - BXB^*$ satisfies the following rank equality

$$r(\, A - BXB^*\,) = 2r[\, A,\, B\,] - r(M) + r[\, F_{S_1}(\, X + S^*M^{\dagger}S\,)F_{S_1}\,], \tag{2.26}$$

where the Moore-Penrose inverse of M in (2.26) can be written as

$$M^{\dagger} = \begin{bmatrix} (E_BAE_B)^{\dagger} & (B^{\dagger})^* - (E_BAE_B)^{\dagger}A(B^{\dagger})^* \\ B^{\dagger} - B^{\dagger}A(E_BAE_B)^{\dagger} & -B^{\dagger}A(B^{\dagger})^* + B^{\dagger}A(E_BAE_B)^{\dagger}A(B^{\dagger})^* \end{bmatrix},$$

see Hall (1975). Thus the rank of $A - BXB^*$ can be written as

$$\begin{aligned} r(\, A - BXB^*\,) &= 2r[\, A,\, B\,] - r(M) \\ &\quad + r\{\, F_{S_1}[\, X - B^{\dagger}A(B^{\dagger})^* + B^{\dagger}A(E_BAE_B)^{\dagger}A(B^{\dagger})^*\,]F_{S_1}\,\}. \end{aligned} \tag{2.27}$$

It is easy to verify by Lemma 2.2(a) that under the condition $A \geq 0$, both $E_BAE_B \geq 0$ and $(E_BAE_B)^{\dagger} \geq 0$ hold. In this case, we can also derive from Lemmas 2.2(a) and 2.3(a) that

$$\begin{aligned} \begin{bmatrix} (E_BAE_B)^{\dagger} & (E_BAE_B)^{\dagger}A \\ A(E_BAE_B)^{\dagger} & A \end{bmatrix} \geq 0 &\Rightarrow A - A(E_BAE_B)^{\dagger}A \geq 0 \\ &\Rightarrow B^{\dagger}A(B^{\dagger})^* - B^{\dagger}A(E_BAE_B)^{\dagger}A(B^{\dagger})^* \geq 0, \end{aligned}$$

that is $B^\dagger A(B^\dagger)^* - B^\dagger A(E_BAE_B)^\dagger A(B^\dagger)^*$ in (2.27) is nonnegative definite. Notice that there always exists a $0 \le U \in \mathbb{C}^{n\times n}$ such that

$$r(F_{S_1}UF_{S_1}) = r(F_{S_1}), \tag{2.28}$$

say, $U = F_{S_1} \ge 0$. In this case, let

$$X = B^\dagger A(B^\dagger)^* - B^\dagger A(E_BAE_B)^\dagger A(B^\dagger)^* + U. \tag{2.29}$$

Then we see from (2.29) that

$$X \ge 0 \ \text{ and } \ r\{\, F_{S_1}[\, X - B^\dagger A(B^\dagger)^* + B^\dagger A(E_BAE_B)^\dagger A(B^\dagger)^* \,]F_{S_1}\,\} = r(F_{S_1}). \tag{2.30}$$

and from (2.18), (2.27) and (2.30) that

$$\begin{aligned} r(\, A - BXB^*\,) &= 2r[\, A,\, B\,] - r(M) + r(F_{S_1}) \\ &= 2r[\, A,\, B\,] - r(M) + n - r(S - MM^\dagger S) \\ &= 2r[\, A,\, B\,] + n - r[\, M,\, S\,] \\ &= 2r[\, A,\, B\,] + n - r\begin{bmatrix} A & B & 0 \\ B^* & 0 & I_n \end{bmatrix} \\ &= r[\, A,\, B\,], \end{aligned}$$

establishing (2.21). Similarly, we can derive from (2.27) that the general expression of $X \ge 0$ satisfying (2.22) can be written as

$$X = B^\dagger A(B^\dagger)^* - B^\dagger A(E_BAE_B)^\dagger A(B^\dagger)^* + S_1^*VV^*S_1,$$

where $V \in \mathbb{C}^{n\times(m+n)}$ is arbitrary. Eqs. (2.23) and (2.24) are obvious. □

3 General solutions to $AX + (AX)^* \ge B$

For a pair of matrices A and B, there may or may not exist solutions to

$$AX + (AX)^* \ge B. \tag{3.1}$$

For example,

$AX + (AX)^* \ge I_3$ has no solution for any A with $r(A) = 1$;

$AX + (AX)^* \ge -I_3$ always has a solution, say, $X = 0$;

$AX + (AX)^* = -I_3$ has no solution for any A with $r(A) = 1$.

These facts indicate that the consistency of the inequality in (3.1) and the consistency of the matrix equation $AX + (AX)^* = B$ are not necessarily equivalent, in particular, the consistency of (3.1) depends on the definiteness of B in (3.1).

Theorem 3.1 *Let $A \in \mathbb{C}^{m\times n}$ and $B = B^* \in \mathbb{C}^{m\times m}$ be given. Then,*

(a) *There exists an $X \in \mathbb{C}^{n\times m}$ that satisfies the following matrix inequality*

$$AX + (AX)^* \geq B \tag{3.2}$$

if and only if

$$E_A B E_A \leq 0 \tag{3.3}$$

holds. In this case, the general solution to (3.2) *can be written as*

$$X = \frac{1}{2}A^{\dagger}[\,B+(\,M+AU\,)(\,M+AU\,)^*\,](\,2I_m - AA^{\dagger}\,) + VA^* + F_A W, \tag{3.4}$$

where $M = (-E_A B E_A)^{1/2}$, and $U, W \in \mathbb{C}^{n\times m}$ and $V = -V^* \in \mathbb{C}^{n\times n}$ are arbitrary.

(b) *There exists an $X \in \mathbb{C}^{n\times m}$ that satisfies the following matrix inequality*

$$AX + (AX)^* > B \tag{3.5}$$

if and only if

$$E_A B E_A \leq 0 \quad \text{and} \quad r(E_A B E_A) = m - r(A) \tag{3.6}$$

hold. In this case, the general solution to (3.5) *can be written as* (3.4), *in which U is any matrix such that $r[\,(-E_A B E_A)^{1/2} + AU\,] = m$, and $W \in \mathbb{C}^{n\times m}$ and $V = -V^* \in \mathbb{C}^{n\times n}$ are arbitrary.*

(c) *There exists an $X \in \mathbb{C}^{n\times m}$ that satisfies the following matrix inequality*

$$AX + (AX)^* \leq B \tag{3.7}$$

if and only if

$$E_A B E_A \geq 0 \tag{3.8}$$

holds. In this case, the general solution to (3.7) *can be written as*

$$X = \frac{1}{2}A^{\dagger}[\,B-(\,M+AU\,)(\,M+AU\,)^*\,](\,2I_m - AA^{\dagger}\,) + VA^* + F_A W, \tag{3.9}$$

where $M = (E_A B E_A)^{1/2}$, and $U, W \in \mathbb{C}^{n\times m}$ and $V = -V^* \in \mathbb{C}^{n\times n}$ are arbitrary.

(d) *There exists an* $X \in \mathbb{C}^{n\times m}$ *that satisfies the following matrix inequality*

$$AX + (AX)^* < B \tag{3.10}$$

if and only if

$$E_ABE_A \geq 0 \quad \text{and} \quad r(E_ABE_A) = m - r(A) \tag{3.11}$$

hold. In this case, the general solution to (3.10) *can be written as* (3.9)*, in which* U *is any matrix such that* $r[\,(E_ABE_A)^{1/2} + AU\,] = m$*, and* $W \in \mathbb{C}^{n\times m}$ *and* $V = -V^* \in \mathbb{C}^{n\times n}$ *are arbitrary.*

Proof. Inequality (3.2) is equivalent to

$$AX + (AX)^* = B + YY^* \tag{3.12}$$

for some Y. From Lemma 2.4(a), the equation is solvable for X if and only if YY^* satisfies

$$E_AYY^*E_A = -E_ABE_A. \tag{3.13}$$

From Lemma 2.6(b), the equation is solvable for Y if and only if (3.3) holds. In this case, the general solution to (3.13) can be written as

$$YY^* = [\,(-E_ABE_A)^{1/2} + AU\,][\,(-E_ABE_A)^{1/2} + AU\,]^*,$$

where U is an arbitrary matrix. Substituting the YY^* into (3.12) gives

$$AX + (AX)^* = B + [\,(-E_ABE_A)^{1/2} + AU\,][\,(-E_ABE_A)^{1/2} + AU\,]^*. \tag{3.14}$$

From Lemma 2.4(a), the general solution to this equation can be written as

$$\begin{aligned} X \;=\; & \frac{1}{2}A^{\dagger}(\,B + [\,(-E_ABE_A)^{1/2} + AU\,][\,(-E_ABE_A)^{1/2} + AU\,]^*\,)(\,2I_m - AA^{\dagger}\,) \\ & + VA^* + F_AW, \end{aligned}$$

where U, $W \in \mathbb{C}^{n\times m}$, and $V = -V^* \in \mathbb{C}^{n\times n}$ are arbitrary, as required for (3.4). It can be seen from (3.14) that (3.5) holds if and only if

$$[\,(-E_ABE_A)^{1/2} + AU\,][\,(-E_ABE_A)^{1/2} + AU\,]^* > 0,$$

that is, $r[\,(-E_ABE_A)^{1/2} + AU\,] = m$. Applying (2.16) and (2.21) gives

$$\begin{aligned} \max_U r[\,(-E_ABE_A)^{1/2} + AU\,] &= r[\,A,\, (-E_ABE_A)^{1/2}\,] \\ &= r[\,A,\, E_ABE_A\,] = r(A) + r(E_ABE_A). \end{aligned}$$

Thus (b) follows. Replacing X with $-X$ and B with $-B$ in (a) and (b) leads to (c) and (d). □

Some consequences of Theorem 3.1 are given below.

Theorem 3.2 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times k}$ be given. Then,*

(a) *The general solution $X \in \mathbb{C}^{n\times m}$ to the following matrix inequality*

$$AX + (AX)^* \geq -BB^* \tag{3.15}$$

can be written as

$$X = \frac{1}{2}A^{\dagger}(\, AUB^* + BU^*A^* + AUU^*A^* \,)(\, 2I_m - AA^{\dagger} \,) + VA^* + F_AW, \tag{3.16}$$

where $U \in \mathbb{C}^{n\times k}$, $V = -V^* \in \mathbb{C}^{n\times n}$ and $W \in \mathbb{C}^{n\times m}$ are arbitrary.

(b) *There exists an $X \in \mathbb{C}^{n\times m}$ that satisfies the following matrix inequality*

$$AX + (AX)^* > -BB^* \tag{3.17}$$

if and only if $r[\, A,\, B \,] = m$ holds. In this case, the general solution to (3.17) *can be written as* (3.16), *in which U is any matrix such that $r(\, B + AU \,) = m$, and $V = -V^* \in \mathbb{C}^{n\times n}$ and $W \in \mathbb{C}^{n\times m}$ are arbitrary.*

(c) *The general solution to*

$$AX + (AX)^* \leq BB^*. \tag{3.18}$$

can be written as

$$X = -\frac{1}{2}A^{\dagger}(\, AUB^* + BU^*A^* + AUU^*A^* \,)(\, 2I_m - AA^{\dagger} \,) + VA^* + F_AW, \tag{3.19}$$

where $U \in \mathbb{C}^{n\times k}$, $V = -V^* \in \mathbb{C}^{n\times n}$ and $W \in \mathbb{C}^{n\times m}$ are arbitrary.

(d) *There exists an $X \in \mathbb{C}^{n\times m}$ that satisfies the following matrix inequality*

$$AX + (AX)^* < BB^* \tag{3.20}$$

if and only if $r[\, A,\, B \,] = m$ holds. In this case, the general solution to (3.20) *can be written as* (3.19), *in which U is any matrix such that $r(\, B + AU \,) = m$, and $V = -V^* \in \mathbb{C}^{n\times n}$ and $W \in \mathbb{C}^{n\times m}$ are arbitrary.*

Corollary 3.3 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times k}$ be given. Then,*

(a) *There exists an $X \in \mathbb{C}^{n\times m}$ that satisfies the following matrix inequality*

$$AX + (AX)^* \geq BB^* \tag{3.21}$$

if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ *holds. In this case, the general solution to* (3.21) *can be written as*

$$X = \frac{1}{2}A^{\dagger}BB^* + UU^*A^* + VA^* + F_AW, \tag{3.22}$$

where $U \in \mathbb{C}^{n\times n}$, $V = -V^* \in \mathbb{C}^{n\times n}$ *and* $W \in \mathbb{C}^{n\times m}$ *are arbitrary.*

(b) *There exists an* X *that satisfies the following matrix inequality*

$$AX + (AX)^* > BB^* \tag{3.23}$$

if and only if both $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $r(A) = m$ *hold. In this case, the general solution to* (3.23) *can be written as* (3.22), *in which* U *is any matrix with* $r(AU) = m$, *and* $V = -V^* \in \mathbb{C}^{n\times n}$ *and* $W \in \mathbb{C}^{n\times m}$ *are arbitrary.*

(c) *There exists an* X *that satisfies the following matrix inequality*

$$AX + (AX)^* + BB^* \leq 0 \tag{3.24}$$

if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ *holds. In this case, the general solution to* (3.24) *can be written as*

$$X = -\frac{1}{2}A^{\dagger}BB^* - UU^*A^* + VA^* + F_AW, \tag{3.25}$$

where $U \in \mathbb{C}^{n\times n}$, $V = -V^* \in \mathbb{C}^{n\times n}$ *and* $W \in \mathbb{C}^{n\times m}$ *are arbitrary.*

(d) *There exists an* X *that satisfies the following matrix inequality*

$$AX + (AX)^* + BB^* < 0 \tag{3.26}$$

if and only if both $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $r(A) = m$ *hold. In this case, the general solution to* (3.26) *can be written as* (3.25), *in which* U *is any matrix with* $r(AU) = m$, *and* $V = -V^* \in \mathbb{C}^{n\times n}$ *and* $W \in \mathbb{C}^{n\times m}$ *are arbitrary.*

The solution in (3.22) can also equivalently be written in other forms, for example,

$$X = \frac{1}{2}A^{-}BB^* + UU^*A^* + VA^* + (\,I_m - A^{-}A\,)W, \tag{3.27}$$

where $U \in \mathbb{C}^{n\times n}$, $V = -V^* \in \mathbb{C}^{n\times n}$ and $W \in \mathbb{C}^{n\times m}$ are arbitrary. In particular, setting $U = 0$, then (3.22) is a general solution to the equality in (3.21).

Corollary 3.4 *Let $B \in \mathbb{C}^{m\times p}$ be given. Then,*

(a) *The general solution $X \in \mathbb{C}^{m\times m}$ to the following inequality*

$$X + X^* \geq BB^* \tag{3.28}$$

can be written as

$$X = \frac{1}{2}BB^* + UU^* + V, \tag{3.29}$$

where $U \in \mathbb{C}^{n\times n}$ and $V = -V^* \in \mathbb{C}^{n\times n}$ are arbitrary.

(b) *The general solution $X \in \mathbb{C}^{m\times m}$ to the following inequality*

$$X + X^* > BB^* \tag{3.30}$$

can be written as (3.29)*, in which $U \in \mathbb{C}^{n\times n}$ with $r(U) = m$ and $V = -V^* \in \mathbb{C}^{n\times n}$ is arbitrary.*

(c) *The general solution $X \in \mathbb{C}^{m\times m}$ to the following inequality*

$$X + X^* \leq BB^* \tag{3.31}$$

can be written as

$$X = \frac{1}{2}BB^* - UU^* + V, \tag{3.32}$$

where $U \in \mathbb{C}^{n\times n}$ and $V = -V^* \in \mathbb{C}^{n\times n}$ are arbitrary.

(d) *The general solution $X \in \mathbb{C}^{m\times m}$ to the following inequality*

$$X + X^* < BB^* \tag{3.33}$$

can be written as (3.32)*, in which $U \in \mathbb{C}^{n\times n}$ with $r(U) = m$ and $V = -V^* \in \mathbb{C}^{n\times n}$ is arbitrary.*

Corollary 3.5 *Let $A \in \mathbb{C}^{m\times n}$ be given. Then,*

(a) *The general solution $X \in \mathbb{C}^{n\times m}$ to the following inequality*

$$AX + (AX)^* \geq 0 \tag{3.34}$$

can be written as

$$X = UU^*A^* + VA^* + F_AW, \tag{3.35}$$

where $U \in \mathbb{C}^{n\times n}$, $V = -V^* \in \mathbb{C}^{n\times n}$ and $W \in \mathbb{C}^{n\times m}$ are arbitrary.

(b) *There exists an* $X \in \mathbb{C}^{n\times m}$ *that satisfies the following matrix inequality*

$$AX + (AX)^* > 0 \tag{3.36}$$

if and only if $r(A) = m$. *In this case, the general solution to* (3.36) *can be written as* (3.35), *in which* $U \in \mathbb{C}^{n\times n}$ *is any matrix with* $r(AU) = m$, $V = -V^* \in \mathbb{C}^{n\times n}$ *and* $W \in \mathbb{C}^{n\times m}$ *are arbitrary.*

(c) *The general solution* $X \in \mathbb{C}^{n\times m}$ *to the following inequality*

$$AX + (AX)^* \leq 0 \tag{3.37}$$

can be written as

$$X = -UU^*A^* + VA^* + F_AW, \tag{3.38}$$

where $U \in \mathbb{C}^{n\times n}$, $V = -V^* \in \mathbb{C}^{n\times n}$ *and* $W \in \mathbb{C}^{n\times m}$ *are arbitrary. In this case,*

$$r[\,AX + (AX)^*\,] = r(AU). \tag{3.39}$$

(d) *There exists an* $X \in \mathbb{C}^{n\times m}$ *that satisfies the following matrix inequality*

$$AX + (AX)^* < 0 \tag{3.40}$$

if and only if $r(A) = m$. *In this case, the general solution to* (3.40) *can be written as* (3.38), *in which* $U \in \mathbb{C}^{n\times n}$ *is any matrix with* $r(AU) = m$, *and* $V = -V^* \in \mathbb{C}^{n\times n}$ *and* $W \in \mathbb{C}^{n\times m}$ *are arbitrary. In particular, if* A *is square and nonsingular, then the general solution to* (3.40) *can be written as*

$$X = -UU^*A^* + VA^*, \tag{3.41}$$

where $U \in \mathbb{C}^{n\times n}$ *is any matrix with* $r(AU) = m$, *and* $V = -V^* \in \mathbb{C}^{n\times n}$ *is arbitrary.*

In Chan and Kwong (1985), solutions to the inequality $(A+B)X+X^*(A+B)^* \geq AB + BA$ were considered for $A \geq 0$ and $B \geq 0$. Applying Theorem 3.1 to this inequality gives the following result.

Corollary 3.6 *Let* A, $B \in \mathbb{C}^{m\times m}$ *be given with* $A \geq 0$ *and* $B \geq 0$. *Then,*

(a) *The general solution to the following matrix inequality*

$$(A + B)X + X^*(A + B)^* \geq AB + BA \tag{3.42}$$

can be written as

$$X = \left(\frac{1}{2}I_m - \frac{1}{2}(A+B)^{\dagger} + UU^* + V\right)(A+B) + F_{(A+B)}W, \tag{3.43}$$

where U, $W \in \mathbb{C}^{m\times m}$ *and* $V = -V^* \in \mathbb{C}^{m\times m}$ *are arbitrary. In particular,*

$$X = \frac{1}{2}(A+B) - \frac{1}{2}(A+B)^{\dagger}(A+B) + V(A+B) + F_{(A+B)}W \tag{3.44}$$

is the general solution to the equality in (3.42).

(b) *There exists an* $X \in \mathbb{C}^{m\times m}$ *that satisfies the following matrix inequality*

$$(A+B)X + X^*(A+B)^* > AB + BA \tag{3.45}$$

if and only if $r(A+B) = m$ *holds. In this case, the general solution to* (3.45) *can be written as*

$$X = \frac{1}{2}(A+B) - \frac{1}{2}I_m + (UU^* + V)(A+B) \tag{3.46}$$

where $U \in \mathbb{C}^{m\times m}$ *is any matrix with* $r(U) = m$, *and* $V = -V^* \in \mathbb{C}^{m\times m}$ *and* $W \in \mathbb{C}^{m\times m}$ *are arbitrary.*

(c) *There always exists an* $XX^* \in \mathbb{C}^{m\times m}$ *that satisfies the following matrix inequality*

$$(A+B)XX^* + XX^*(A+B)^* \geq AB + BA, \tag{3.47}$$

and partial solutions to (3.47) *can be written as*

$$XX^* = \frac{1}{2}(A+B) + k(A+B)^{\dagger}(A+B) + F_{(A+B)}WW^*F_{(A+B)}, \tag{3.48}$$

where $0 \leq k$ *is any real number, and* $W \in \mathbb{C}^{m\times m}$ *is arbitrary.*

(d) *There exists an* $XX^* \in \mathbb{C}^{m\times m}$ *that satisfies the following matrix inequality*

$$(A+B)XX^* + XX^*(A+B)^* > AB + BA \tag{3.49}$$

if and only if $r(A+B) = m$ *holds. In this case, the general solution to* (3.49) *can be written as*

$$X = \frac{1}{2}(A+B) + kI_m, \tag{3.50}$$

where $0 \leq k$ *is any real number.*

4 General solutions to the inequality $AXA^* \geq B$

In this section, we consider the following linear matrix inequality

$$AXA^* \geq B \tag{4.1}$$

and its variations, where $A \in \mathbb{C}^{m\times n}$ and $B = B^* \in \mathbb{C}^{m\times m}$ are given. The linear matrix equation corresponding to (4.1) is $AXA^* = B$ given in Lemma 2.6.

It should be pointed out that for any two Hermitian matrices P and Q, the inequality $P \geq Q$ does not imply that $r(Q) \leq r(P)$ or $\mathscr{R}(Q) \subseteq \mathscr{R}(P)$, for example, Let $P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $P - Q = I_2 > 0$. Hence (4.1) does not imply that $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ for a general Hermitian matrix B, although the equality in (4.1) holds if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$.

Theorem 4.1 *Let $A \in \mathbb{C}^{m\times n}$ and $B = B^* \in \mathbb{C}^{m\times m}$ be given. Then,*

(a) *There exists an $X \in \mathbb{C}^{n\times n}$ such that* (4.1) *holds if and only if*

$$E_ABE_A \leq 0 \quad \textit{and} \quad r(E_ABE_A) = r(E_AB). \tag{4.2}$$

In this case, the general solution to (4.1) *can be written as*

$$\begin{aligned} X \;=\; & A^{\dagger}B(A^{\dagger})^* - A^{\dagger}BE_A(E_ABE_A)^{\dagger}E_AB(A^{\dagger})^* \\ & + A^{\dagger}UU^*(A^{\dagger})^* + W - A^{\dagger}AWA^{\dagger}A, \end{aligned} \tag{4.3}$$

where $U \in \mathbb{C}^{m\times m}$ and $W \in \mathbb{C}^{n\times n}$ are arbitrary. In particular, if $W = W^*$, then the X in (4.3) *satisfies $X = X^*$.*

(b) *There exists an $X \in \mathbb{C}^{n\times n}$ such that*

$$AXA^* > B \tag{4.4}$$

holds if and only if

$$E_ABE_A \leq 0 \quad \textit{and} \quad r(E_ABE_A) = r(E_A). \tag{4.5}$$

In this case, the general solution to (4.4) *can be written as* (4.3), *in which U is any matrix such that $r[\, BE_A(E_ABE_A)^{\dagger}E_AB - AA^{\dagger}UU^*AA^{\dagger} \,] = m$ hold, and $W \in \mathbb{C}^{n\times n}$ is arbitrary.*

(c) *There exists an* $X \in \mathbb{C}^{n\times n}$ *such that*

$$AXA^* + B \le 0 \tag{4.6}$$

holds if and only if (4.2) *holds. In this case, the general solution to* (4.6) *can be written as*

$$\begin{aligned} X &= -A^\dagger B(A^\dagger)^* + A^\dagger BE_A(E_ABE_A)^\dagger E_AB(A^\dagger)^* - A^\dagger UU^*(A^\dagger)^* \\ &\quad + W - A^\dagger AWA^\dagger A, \end{aligned} \tag{4.7}$$

where $U \in \mathbb{C}^{m\times m}$ *and* $W \in \mathbb{C}^{n\times n}$ *are arbitrary.*

(d) *There exists an* $X \in \mathbb{C}^{n\times n}$ *such that*

$$AXA^* + B < 0 \tag{4.8}$$

holds if and only if (4.5) *holds. In this case, the general solution to* (4.8) *can be written as* (4.7), *in which* U *is any matrix such that* $r[\, BE_A(E_ABE_A)^\dagger E_AB - AA^\dagger UU^*AA^\dagger \,] = m$ *holds, and* $W \in \mathbb{C}^{n\times n}$ *is arbitrary.*

Proof. Inequality (4.1) is equivalent to

$$AXA^* = B + YY^* \tag{4.9}$$

for some Y. From Lemma 2.6(a), (4.9) is solvable for X if and only if $E_A(B + YY^*) = 0$, that is,

$$E_AYY^* = -E_AB. \tag{4.10}$$

From Lemma 2.5(b), (4.10) is solvable for YY^* if and only if $E_ABE_A \le 0$ and $r(E_ABE_A) = r(E_AB)$, establishing (4.2). In this case, the general solution to (4.10) can be written as

$$YY^* = -BE_A(E_ABE_A)^\dagger E_AB + AA^\dagger UU^*AA^\dagger,$$

where U is an arbitrary matrix. Substituting the YY^* into (4.9) gives

$$AXA^* = B - BE_A(E_ABE_A)^\dagger E_AB + AA^\dagger UU^*AA^\dagger. \tag{4.11}$$

From Lemma 2.6(a), the general solution to (4.11) can be written as

$$X = A^\dagger B(A^\dagger)^* - A^\dagger BE_A(E_ABE_A)^\dagger E_AB(A^\dagger)^* + A^\dagger UU^*(A^\dagger)^* + W - A^\dagger AWA^\dagger A,$$

where $U \in \mathbb{C}^{m\times m}$ and $W \in \mathbb{C}^{n\times n}$ are arbitrary, as required for (4.3). It can be seen from (4.11) that (4.4) holds if and only if

$$-BE_A(E_ABE_A)^{\dagger}E_AB + AA^{\dagger}UU^*AA^{\dagger} > 0. \tag{4.12}$$

Applying (2.15) and (2.23) gives

$$\begin{aligned}
&\max_{UU^*} r[\,-BE_A(E_ABE_A)^{\dagger}E_AB + AA^{\dagger}UU^*AA^{\dagger}\,]\\
&\quad = r[\,-BE_A(E_ABE_A)^{\dagger}E_AB,\ AA^{\dagger}\,]\\
&\quad = r[\,BE_A,\ A\,]\\
&\quad = r(E_ABE_A) + r(A),
\end{aligned}$$

so that (4.4) holds if and only if $r(E_ABE_A) + r(A) = m$. Thus (b) follows. Replacing X with $-X$ in (a) and (b) leads to (c) and (d). □

Some consequences of Theorem 4.1 are give below.

Theorem 4.2 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times k}$ be given. Then,*

(a) *The inequality*

$$AXA^* \geq -BB^* \tag{4.13}$$

always has solution, and the general solution to (4.13) *can be written as*

$$\begin{aligned} X \;=\;& A^{\dagger}B(E_AB)^{\dagger}(E_AB)B^*(A^{\dagger})^* - A^{\dagger}BB^*(A^{\dagger})^* + A^{\dagger}UU^*(A^{\dagger})^*\\ &+W - A^{\dagger}AWA^{\dagger}A, \end{aligned} \tag{4.14}$$

where $U \in \mathbb{C}^{m\times m}$ and $W \in \mathbb{C}^{n\times n}$ are arbitrary.

(b) *There exists an $X \in \mathbb{C}^{n\times n}$ such that*

$$AXA^* > -BB^* \tag{4.15}$$

holds if and only if $r[\,A,\ B\,] = m$. In this case, the general solution to (4.15) *can be written as* (4.14)*, in which U is any matrix such that*

$$r[\,B(E_AB)^{\dagger}(E_AB)B^* + AA^{\dagger}UU^*AA^{\dagger}\,] = m,$$

and $W \in \mathbb{C}^{n\times n}$ is arbitrary.

(c) *The inequality*

$$AXA^* \leq BB^* \tag{4.16}$$

always has solution, and the general solution to (4.16) *can be written as*

$$\begin{aligned} X \;=\;& A^{\dagger}BB^*(A^{\dagger})^* - A^{\dagger}B(E_AB)^{\dagger}(E_AB)B^*(A^{\dagger})^* - A^{\dagger}UU^*(A^{\dagger})^*\\ &+W - A^{\dagger}AWA^{\dagger}A, \end{aligned} \tag{4.17}$$

where $U \in \mathbb{C}^{m\times m}$ and $W \in \mathbb{C}^{n\times n}$ are arbitrary.

(d) *There exists an* $X \in \mathbb{C}^{n\times n}$ *such that*

$$AXA^* < BB^* \tag{4.18}$$

holds if and only if $r[\, A,\, B\,] = m$. *In this case, the general solution to* (4.18) *can be written as* (4.17), *in which* U *is any matrix such that* $r[\, B(E_AB)^{\dagger}(E_AB)B^* + AA^{\dagger}UU^*AA^{\dagger}\,] = m$, *and* $W \in \mathbb{C}^{n\times n}$ *is arbitrary.*

Corollary 4.3 *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{m\times p}$ *be given. Then,*

(a) *There exists an* $X \in \mathbb{C}^{n\times n}$ *such that*

$$AXA^* \geq BB^* \tag{4.19}$$

holds if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$. *In this case, the general solution to* (4.1) *can be written as*

$$X = A^{\dagger}BB^*(A^{\dagger})^* + UU^* + W - A^{\dagger}AWA^{\dagger}A, \tag{4.20}$$

where $U,\, W \in \mathbb{C}^{n\times n}$ *are arbitrary.*

(b) *There exists an* $X \in \mathbb{C}^{n\times n}$ *such that*

$$AXA^* > BB^* \tag{4.21}$$

holds if and only if $r(A) = m$. *In this case, the general solution to* (4.21) *can be written as* (4.20), *in which* $U \in \mathbb{C}^{n\times n}$ *is any matrix with* $r(U) = m$, *and* $W \in \mathbb{C}^{n\times n}$ *is arbitrary.*

(c) *There exists an* $X \in \mathbb{C}^{n\times n}$ *such that*

$$AXA^* + BB^* \leq 0 \tag{4.22}$$

holds if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$. *In this case, the general solution to* (4.1) *can be written as*

$$X = -A^{\dagger}BB^*(A^{\dagger})^* - UU^* + W - A^{\dagger}AWA^{\dagger}A, \tag{4.23}$$

where $U,\, W \in \mathbb{C}^{n\times n}$ *are arbitrary.*

(d) *There exists an* $X \in \mathbb{C}^{n\times n}$ *such that*

$$AXA^* + BB^* < 0 \tag{4.24}$$

holds if and only if $r(A) = m$. *In this case, the general solution to* (4.24) *can be written as* (4.23), *in which* $U \in \mathbb{C}^{n\times n}$ *is any matrix with* $r(U) = m$, *and* $W \in \mathbb{C}^{n\times n}$ *is arbitrary.*

Corollary 4.4 *Let $A \in \mathbb{C}^{m\times n}$ be given. Then,*

(a) *The general solution to the inequality $AXA^* \geq 0$ can be written as*

$$X = UU^* + W - A^\dagger AWA^\dagger A, \tag{4.25}$$

where $U, W \in \mathbb{C}^{n\times n}$ are arbitrary.

(b) *There exists an X such that $AXA^* > 0$ holds if and only if $r(A) = m$. In this case, the general solution to $AXA^* > 0$ can be written as (4.25), in which $U, W \in \mathbb{C}^{n\times n}$ are arbitrary.*

(c) *The general solution to the inequality $AXA^* \leq 0$ can be written as*

$$X = -UU^* + W - A^\dagger AWA^\dagger A, \tag{4.26}$$

where $U, W \in \mathbb{C}^{n\times n}$ are arbitrary.

(d) *There exists an X such that $AXA^* < 0$ holds if and only if $r(A) = m$. In this case, the general solution to $AXA^* < 0$ can be written as (4.26), in which $U \in \mathbb{C}^{n\times n}$ is any matrix with $r(U) = m$, and $W \in \mathbb{C}^{n\times n}$ is arbitrary.*

The following theorem is derived from Lemma 2.6(b), and its proof is omitted.

Theorem 4.5 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times k}$ be given. Then,*

(a) *There exists an X such that*

$$AXX^*A^* \geq BB^* \tag{4.27}$$

holds if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$. In this case, the general solution to (4.27) can be written as

$$XX^* = (\, A^\dagger B + E_AW \,)(\, A^\dagger B + E_AW \,)^* + UU^*, \tag{4.28}$$

where $U \in \mathbb{C}^{n\times n}$ and $W \in \mathbb{C}^{n\times k}$ are arbitrary.

(b) *There exists an X such that*

$$AXX^*A^* > BB^* \tag{4.29}$$

holds if and only if $r(A) = m$. In this case, the general solution to (4.29) can be written as (4.28), in which, $U \in \mathbb{C}^{n\times n}$ is any matrix with $r(U) = m$ and $W \in \mathbb{C}^{n\times k}$ is arbitrary.

An application to partitioned matrices is given below.

Corollary 4.6 *Let*

$$M(X) = \begin{bmatrix} AXA^* & B \\ B^* & CC^* \end{bmatrix}, \tag{4.30}$$

where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$ *and* $C \in \mathbb{C}^{p\times q}$ *are given. Then,*

(a) *There exists an* X *such that* $M(X) \geq 0$ *in* (4.30) *if and only if*

$$\mathscr{R}(B) \subseteq \mathscr{R}(A) \ \ \text{and} \ \ \mathscr{R}(B^*) \subseteq \mathscr{R}(C). \tag{4.31}$$

In this case, the general solution to $M \geq 0$ *can be written as*

$$X = A^{\dagger}B(CC^*)^{\dagger}B^*(A^{\dagger})^* + UU^* + W - A^{\dagger}AWA^{\dagger}A, \tag{4.32}$$

where $U, \ W \in \mathbb{C}^{n\times n}$ *are arbitrary.*

(b) *There exists an* X *such that* $M > 0$ *in* (4.30) *if and only if*

$$r(A) = m \ \ \text{and} \ \ r(C) = p. \tag{4.33}$$

In this case, the general solution to $M > 0$ *can be written as*

$$X = A^{\dagger}B(CC^*)^{-1}B^*(A^{\dagger})^* + UU^*, \tag{4.34}$$

where $U \in \mathbb{C}^{n\times n}$ *is any matrix with* $r(U) = m$ *and* $W \in \mathbb{C}^{n\times n}$ *is arbitrary.*

Proof. It is easily seen from Lemma 2.3 that

$$M \geq 0 \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A), \ \ \mathscr{R}(B^*) \subseteq \mathscr{R}(C) \text{ and } AXA^* \geq B(CC^*)^{+}B^*, \tag{4.35}$$

$$M > 0 \Leftrightarrow r(A) = m, \ \ r(C) = p \text{ and } AXA^* > B(CC^*)^{+}B^*. \tag{4.36}$$

Solving the two inequalities in (4.35) and (4.36) by Theorem 3.1 leads to (a) and (b). □

A challenging problem is to give the general solution to the inequality $AXX^*A^* \leq BB^*$. It is obvious that the inequality has a trivial solution $X = 0$. However, the inequality may have only zero solution in some cases. For example, the inequality

$$\begin{bmatrix} x^2 & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

only has a solution $x = 0$. The following result gives the identifying conditions for $AXX^*A^* \leq BB^*$ to have nonzero solutions and their general expressions.

Theorem 4.7 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times k}$ be given. Then,*

(a) *There exists a matrix X such that both $AX \neq 0$ and*

$$AXX^*A^* \leq BB^* \tag{4.37}$$

hold if and only if

$$\mathscr{R}(A) \cap \mathscr{R}(B) \neq \{0\}. \tag{4.38}$$

In this case, a solution to (4.37) *can be written as*

$$XX^* = A^\dagger BF_{B_1}VF_{B_1}B^*(A^\dagger)^* + E_AWW^*E_A, \tag{4.39}$$

where $B_1 = E_AB$, $V \in \mathbb{C}^{k\times k}$ is any matrix with $0 < V \leq I_k$, and $W \in \mathbb{C}^{n\times n}$ is arbitrary.

(b) *There exists a matrix X such that $AX \neq 0$ and*

$$AXX^*A^* < BB^* \tag{4.40}$$

hold if and only if

$$A \neq 0 \quad \text{and} \quad r(B) = m. \tag{4.41}$$

In this case, a solution to (4.40) *can be written as* (4.39), *in which $V \in \mathbb{C}^{k\times k}$ is any matrix with $0 < V < I_k$, and $W \in \mathbb{C}^{n\times n}$ is arbitrary.*

(c) *Under the condition $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, there always exists a matrix X such that both $AX \neq 0$ and*

$$AXX^*A^* \leq BB^* \tag{4.42}$$

hold, and a solution to (4.42) *can be written as*

$$XX^* = A^\dagger BVB^*(A^\dagger)^* + E_AWW^*E_A, \tag{4.43}$$

where $V \in \mathbb{C}^{k\times k}$ is any matrix with $0 < V \leq I_k$, and $W \in \mathbb{C}^{n\times n}$ is arbitrary.

(d) *Under the condition $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, there exists a matrix X such that $AX \neq 0$ and*

$$AXX^*A^* < BB^* \tag{4.44}$$

hold if and only if $r(B) = m$. In this case, the general solution to (4.44) *can be written as* (4.43), *in which $V \in \mathbb{C}^{k\times k}$ is any matrix with $0 < V < I_k$, and $W \in \mathbb{C}^{n\times n}$ is arbitrary.*

Proof. It can be seen from Lemma 2.1 that if there exists an X such that $AX \neq 0$ and (4.37) hold, then $\mathscr{R}(AX) \subseteq \mathscr{R}(B)$ holds, which obviously implies that (4.38) holds. On the other hand, it can be derived from $E_ABF_{E_AB} = 0$ that

$$AA^{\dagger}BF_{E_AB} = BF_{E_AB}, \tag{4.45}$$

and from Lemma 2.8 that

$$r(BF_{E_AB}) = r\begin{bmatrix} B \\ E_AB \end{bmatrix} - r(E_AB) = r(A) + r(B) - r[\, A,\, B\,] = \dim[\mathscr{R}(A) \cap \mathscr{R}(B)]. \tag{4.46}$$

Hence if (4.38) holds, then $BF_{E_AB} \neq 0$ and $\mathscr{R}(BF_{E_AB}) = \mathscr{R}(A) \cap \mathscr{R}(B)$ by (4.45) and (4.46). In this case,

$$AA^{\dagger}BF_{E_AB}VF_{E_AB}B^*(A^{\dagger})^*A = BF_{E_AB}VF_{E_AB}B^*.$$

Thus we can derive from (4.39) and Lemma 2.2(a) that

$$BB^* - AXX^*A^* = BB^* - BF_{E_AB}VF_{E_AB}B^* = B(\, I_k - F_{E_AB}VF_{E_AB}\,)B^* \geq 0,$$

that is, (4.39) is a solution to (4.37). The two conditions in (4.41) are obvious under the condition that both $AX \neq 0$ and (4.40) hold. Conversely, if (4.41) holds, we can derive from (4.40) and Lemma 2.2(b) that $I_k - F_{E_AB}VF_{E_AB} > 0$ and

$$BB^* - AXX^*A^* = BB^* - BF_{E_AB}VF_{E_AB}B^* = B(\, I_k - F_{E_AB}VF_{E_AB}\,)B^* > 0.$$

Results (c) and (d) are direct consequences of (a) and (b). □

5 General solutions to the inequality $AXB \geq C$

Theorem 5.1 *Let $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ and $C = C^* \in \mathbb{C}^{m\times m}$ be given, and denote $M = [\, E_A,\, F_B\,]$. Then,*

(a) *There exists an $X \in \mathbb{C}^{p\times q}$ that satisfies the following matrix inequality*

$$AXB \geq C \tag{5.1}$$

if and only if

$$M^*CM \leq 0 \quad \text{and} \quad r(M^*CM) = r(CM) \tag{5.2}$$

holds, or equivalently,

$$(2I_m - AA^{\dagger} - B^{\dagger}B)C(2I_m - AA^{\dagger} - B^{\dagger}B) \leq 0 \tag{5.3}$$

and

$$r\begin{bmatrix} C & C & A & 0 \\ C & C & 0 & B^* \\ A^* & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{bmatrix} = r\begin{bmatrix} C & A & 0 \\ C & 0 & B^* \end{bmatrix} + r(A) + r(B). \tag{5.4}$$

In this case, the general solution to (5.1) *can be written as*

$$\begin{aligned} X &= A^{\dagger}CB^{\dagger} - A^{\dagger}CM(M^*CM)^{\dagger}M^*CB^{\dagger} + A^{\dagger}E_MUU^*E_MB^{\dagger} \\ &\quad + W - A^{\dagger}AWBB^{\dagger}, \end{aligned} \tag{5.5}$$

where $U \in \mathbb{C}^{m\times m}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(b) *There exists an* X *that satisfies the following matrix inequality*

$$AXB > C \tag{5.6}$$

if and only if

$$M^*CM \leq 0 \ \ \textit{and} \ \ r\begin{bmatrix} C & A & 0 \\ C & 0 & B^* \end{bmatrix} = m + r[\, A, \, B^* \,]. \tag{5.7}$$

In this case, the general solution to (5.6) *can be written as* (5.5), *in which* $U \in \mathbb{C}^{m\times m}$ *is any matrix such that* $r[-CM(M^*CM)^{\dagger}M^*C + E_MUU^*E_M\,] = m$, *and* $W \in \mathbb{C}^{p\times q}$ *is arbitrary.*

(c) *There exists an* X *that satisfies*

$$AXB + C \leq 0 \tag{5.8}$$

if and only if (5.2) *holds. In this case, the general solution to* (5.8) *can be written as*

$$\begin{aligned} X &= -A^{\dagger}CB^{\dagger} + A^{\dagger}CM(M^*CM)^{\dagger}M^*CB^{\dagger} - A^{\dagger}E_MUU^*E_MB^{\dagger} \\ &\quad + W - A^{\dagger}AWBB^{\dagger}, \end{aligned} \tag{5.9}$$

where $U \in \mathbb{C}^{m\times m}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(d) *There exists an* X *that satisfies the following matrix inequality*

$$AXB + C < 0 \tag{5.10}$$

if and only if (5.7) *holds. In this case, the general solution to* (5.10) *can be written as* (5.9), *in which* U *is any matrix such that* $r[-CM(M^*CM)^{\dagger}M^*C + E_MUU^*E_M\,] = m$, *and* $W \in \mathbb{C}^{p\times q}$ *is arbitrary.*

Proof. Inequality (5.1) is equivalent to

$$AXB = C + YY^* \tag{5.11}$$

for some Y. From Lemma 2.7, the equation is solvable for X if and only if

$$E_A YY^* = -E_A C \quad \text{and} \quad F_B YY^* = -F_B C, \tag{5.12}$$

this is,

$$\begin{bmatrix} E_A \\ F_B \end{bmatrix} YY^* = -\begin{bmatrix} E_A C \\ F_B C \end{bmatrix}. \tag{5.13}$$

From Lemma 2.5(b), (5.13) is solvable for YY^* if and only if

$$\begin{bmatrix} E_A \\ F_B \end{bmatrix} C[\, E_A,\, F_B\,] \leq 0 \quad \text{and} \quad r\left(\begin{bmatrix} E_A \\ F_B \end{bmatrix} C[\, E_A,\, F_B\,] \right) = r(C[\, E_A,\, F_B\,]),$$

establishing (5.2). The equivalence of (5.2) with (5.3) and (5.4) are derived from Lemma 2.8. Under (5.2), the general solution to (5.11) can be written as

$$YY^* = -CM(M^*CM)^{\dagger}M^*C + E_M UU^* E_M,$$

where U is an arbitrary matrix. Substituting the YY^* into (5.11) gives

$$AXB = C - CM(M^*CM)^{\dagger}M^*C + E_M UU^* E_M. \tag{5.14}$$

From Lemma 2.7, the general solution to (5.14) is

$$X = A^{\dagger}CB^{\dagger} - A^{\dagger}CM(M^*CM)^{\dagger}M^*CB^{\dagger} + A^{\dagger}E_M UU^* E_M B^{\dagger} + W - A^{\dagger}AWBB^{\dagger},$$

establishing (5.5). It can be seen from (5.13) that (5.6) holds if and only if

$$-CM(M^*CM)^{\dagger}M^*C + E_M UU^* E_M > 0.$$

Applying (2.23) gives

$$\begin{aligned} \max_{UU^*} r[\, -CM(M^*CM)^{\dagger}M^*C + E_M UU^* E_M\,] \\ &= r[\, -CM(M^*CM)^{\dagger}M^*C,\, E_M\,] \\ &= r[\, CM,\, E_M\,] \\ &= r(MM^{\dagger}CM) + r(E_M) \\ &= r(CM) + m - r(M) \\ &= r\begin{bmatrix} C & A & 0 \\ C & 0 & B^* \end{bmatrix} - r[\, A,\, B^*\,]. \end{aligned}$$

Thus (b) follows. Replacing X with $-X$ in (a) and (b) leads to (c) and (d). □

Corollary 5.2 *Let* $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ *and* $C =\in \mathbb{C}^{m\times k}$ *be given, and denote* $M = [\, E_A, \, F_B \,]$. *Then,*

(a) *The general solution to*

$$AXB + CC^* \geq 0 \tag{5.15}$$

can be written as

$$\begin{aligned} X &= -A^{\dagger}CC^*B^{\dagger} + A^{\dagger}C(M^*C)^{\dagger}(M^*C)C^*B^{\dagger} + A^{\dagger}E_MUU^*E_MB^{\dagger} \\ &\quad +W - A^{\dagger}AWBB^{\dagger}, \end{aligned} \tag{5.16}$$

where $U \in \mathbb{C}^{m\times m}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(b) *There exists an* X *that satisfies the following matrix inequality*

$$AXB + CC^* > 0 \tag{5.17}$$

if and only if $r\begin{bmatrix} C & A & 0 \\ C & 0 & B^* \end{bmatrix} = m + r[\, A, \, B^* \,]$ *holds. In this case, the general solution to* (5.17) *can be written as* (5.16), *in which* $U \in \mathbb{C}^{m\times m}$ *is any matrix such that* $r[C(M^*C)^{\dagger}(M^*C)C^* + E_MUU^*E_M \,] = m$, *and* $W \in \mathbb{C}^{p\times q}$ *is arbitrary.*

(c) *The general solution to*

$$AXB \leq CC^* \tag{5.18}$$

can be written as

$$\begin{aligned} X &= A^{\dagger}CC^*B^{\dagger} - A^{\dagger}C(M^*C)^{\dagger}(M^*C)C^*B^{\dagger} - A^{\dagger}E_MUU^*E_MB^{\dagger} \\ &\quad +W - A^{\dagger}AWBB^{\dagger}, \end{aligned} \tag{5.19}$$

where $U \in \mathbb{C}^{m\times m}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(d) *There exists an* X *that satisfies the following matrix inequality*

$$AXB < CC^* \tag{5.20}$$

if and only if $r\begin{bmatrix} C & A & 0 \\ C & 0 & B^* \end{bmatrix} = m + r[\, A, \, B^* \,]$ *holds. In this case, the general solution to* (5.20) *can be written as* (5.19), *in which* U *is any matrix such that* $r[C(M^*C)^{\dagger}(M^*C)C^* + E_MUU^*E_M \,] = m$, *and* $W \in \mathbb{C}^{p\times q}$ *is arbitrary.*

Corollary 5.3 *Let $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ and $C \in \mathbb{C}^{m\times k}$ be given, and and denote $M = [\, E_A,\, F_B \,]$. Then,*

(a) *There exists an $X \in \mathbb{C}^{p\times q}$ that satisfies*

$$AXB \geq CC^* \tag{5.21}$$

if and only if

$$\mathscr{R}(C) \subseteq \mathscr{R}(A) \ \ \text{and} \ \ \mathscr{R}(C) \subseteq \mathscr{R}(B^*). \tag{5.22}$$

In this case, the general solution to (5.21) *can be written as*

$$X = A^{\dagger}CC^*B^{\dagger} + A^{\dagger}E_MUU^*E_MB^{\dagger} + W - A^{\dagger}AWBB^{\dagger}, \tag{5.23}$$

where $U \in \mathbb{C}^{m\times m}$ and $W \in \mathbb{C}^{p\times q}$ are arbitrary.

(b) *There exists an X such that*

$$AXB > CC^* \tag{5.24}$$

if and only if $r(A) = r(B) = m$. In this case, the general solution to (5.24) *can be written as* (5.23), *in which $U \in \mathbb{C}^{q\times q}$ is any matrix with $r(E_MU) = m$, and $W \in \mathbb{C}^{p\times q}$ is arbitrary.*

(c) *There exists an $X \in \mathbb{C}^{p\times q}$ that satisfies*

$$AXB + CC^* \leq 0 \tag{5.25}$$

if and only if (5.22) *holds. In this case, the general solution to* (5.25) *can be written as*

$$X = -A^{\dagger}CC^*B^{\dagger} - A^{\dagger}E_MUU^*E_MB^{\dagger} + W - A^{\dagger}AWBB^{\dagger}, \tag{5.26}$$

where $U \in \mathbb{C}^{m\times m}$ and $W \in \mathbb{C}^{p\times q}$ are arbitrary.

(d) *There exists an $X \in \mathbb{C}^{p\times q}$ such that*

$$AXB + CC^* < 0 \tag{5.27}$$

if and only if $r(A) = r(B) = m$. In this case, the general solution to (5.27) *can be written as* (5.26), *in which $U \in \mathbb{C}^{q\times q}$ is any matrix with $r(E_MU) = m$, and $W \in \mathbb{C}^{p\times q}$ is arbitrary.*

Corollary 5.4 *Let $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ and $C \in \mathbb{C}^{m\times k}$ be given, and denote $M = [\, E_A,\, F_B\,]$. Then,*

(a) *The general solution to*

$$AXB \geq 0 \tag{5.28}$$

can be written as

$$X = A^{\dagger}E_MUU^*E_MB^{\dagger} + W - A^{\dagger}AWBB^{\dagger}, \tag{5.29}$$

where $U \in \mathbb{C}^{m\times m}$ and $W \in \mathbb{C}^{p\times q}$ are arbitrary.

(b) *There exists an $X \in \mathbb{C}^{p\times q}$ such that*

$$AXB > 0 \tag{5.30}$$

if and only if $r(A) = r(B) = m$. In this case, the general solution to (5.30) *can be written as* (5.29), *in which $U \in \mathbb{C}^{q\times q}$ is is any matrix with $r(E_MU) = m$, and $W \in \mathbb{C}^{p\times q}$ is arbitrary.*

6 General solutions to the inequality $AXB + (AXB)^* \geq C$

An extension of the matrix equation in (2.1) is given by

$$AXB + (AXB)^* = C, \tag{6.1}$$

where $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ and $C = C^* \in \mathbb{C}^{m\times m}$ are given. This equation was recently considered by Tian and Liu (2006) and the following result was given.

Lemma 6.1 (Tian and Liu, 2006) *Let $G = [\, A,\, B^*\,]$ and $H = [\, B^*,\, A\,]^*$. Then,*

(a) *There exists an $X \in \mathbb{C}^{p\times q}$ such that* (6.1) *holds if and only if*

$$\mathscr{R}(C) \subseteq \mathscr{R}[\, A,\, B^*\,], \quad r\begin{bmatrix} C & A \\ A^* & 0 \end{bmatrix} = 2r(A), \quad r\begin{bmatrix} C & B^* \\ B & 0 \end{bmatrix} = 2r(B), \tag{6.2}$$

or equivalently,

$$[\, A,\, B^*\,][\, A,\, B^*\,]^{\dagger}C = C, \quad E_ACE_A = 0, \quad F_BCF_B = 0. \tag{6.3}$$

Under (6.2), *the general solution to* (6.1) *can be written as*

$$X = \frac{1}{2}(\, Z_1 + Z_2^*\,), \tag{6.4}$$

where Z_1 *and* Z_2 *are the general solutions of the equation* $AZ_1B + B^*Z_2A^* = C$. *Written in an explicit form,*

$$\begin{aligned} X &= X_0 + [\, I_p,\, 0\,]F_G V E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} - [\, 0,\, I_p\,]E_H V^* F_G \begin{bmatrix} 0 \\ I_q \end{bmatrix} \\ &\quad + W - A^\dagger A W B B^\dagger, \end{aligned} \tag{6.5}$$

where X_0 *is a special solution to* (6.1), *and* $V \in \mathbb{C}^{(p+q)\times(p+q)}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(b) *There exists an* X *such that*

$$AXB + (AXB)^* = CC^* \tag{6.6}$$

holds if and only if

$$\mathscr{R}(C) \subseteq \mathscr{R}(A) \quad \text{and} \quad \mathscr{R}(C) \subseteq \mathscr{R}(B^*). \tag{6.7}$$

Under (6.7), *the general solution to* (6.6) *can be written as*

$$\begin{aligned} X &= \frac{1}{2}A^\dagger CC^* B^\dagger + [\, I_p,\, 0\,]F_G V E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} - [\, 0,\, I_p\,]E_H V^* F_G \begin{bmatrix} 0 \\ I_q \end{bmatrix} \\ &\quad + W - A^\dagger A W B B^\dagger, \end{aligned} \tag{6.8}$$

where $V \in \mathbb{C}^{(p+q)\times(p+q)}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(c) *There exists an* X *such that*

$$AXB + (AXB)^* = -CC^* \tag{6.9}$$

holds if and only if

$$\mathscr{R}(C) \subseteq \mathscr{R}(A) \quad \text{and} \quad \mathscr{R}(C) \subseteq \mathscr{R}(B^*). \tag{6.10}$$

Under (6.10), *the general solution to* (6.9) *can be written as*

$$\begin{aligned} X &= -\frac{1}{2}A^\dagger CC^* B^\dagger + [\, I_p,\, 0\,]F_G V E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} - [\, 0,\, I_p\,]E_H V^* F_G \begin{bmatrix} 0 \\ I_q \end{bmatrix} \\ &\quad + W - A^\dagger A W B B^\dagger, \end{aligned} \tag{6.11}$$

where $V \in \mathbb{C}^{(p+q)\times(p+q)}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

A matrix inequality corresponding to (6.1) in Löwner partial ordering is

$$AXB + (AXB)^* \geq C. \tag{6.12}$$

This inequality is equivalent to

$$AXB + (AXB)^* = C + YY^* \tag{6.13}$$

for some Y. From Lemma 6.1(a), the equation is solvable for X if and only if

$$E_M YY^* = -E_M C, \quad E_A YY^* E_A = -E_A C E_A, \quad F_B YY^* F_B = -F_B C F_B, \tag{6.14}$$

where $M = [\, A, \, B^* \,]$. However, we do not know how to solve YY^* from this system of equations for a general Hermitian matrix C, and therefore, we do not know how to solve X in (6.12) for a general Hermitian matrix C. In this section, we only consider a special case of (6.12):

$$AXB + (AXB)^* \geq CC^* \tag{6.15}$$

and its variations.

Theorem 6.2 *Let $A \in \mathbb{C}^{m\times p}$ and $B \in \mathbb{C}^{q\times m}$ be given, and denote $M = [\, E_A, \, F_B \,]$, $G = [\, A, \, B^* \,]$ and $H = [\, B^*, \, A \,]^*$. Then,*

(a) *There exists an $X \in \mathbb{C}^{p\times q}$ that satisfies (6.15) if and only if (6.7) holds. In this case, the general solution to (6.15) can be written as*

$$X = \frac{1}{2}A^\dagger CC^* B^\dagger + A^\dagger E_M UU^* E_M B^\dagger + [\, I_p, \, 0 \,] F_G V E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} - [\, 0, \, I_p \,] E_H V^* F_G \begin{bmatrix} 0 \\ I_q \end{bmatrix} + W - A^\dagger A W B B^\dagger, \tag{6.16}$$

where $U \in \mathbb{C}^{m\times m}$, and $V \in \mathbb{C}^{(p+q)\times(p+q)}$ and $W \in \mathbb{C}^{p\times q}$ are arbitrary.

(b) *There exists an X that satisfies the following matrix inequality*

$$AXB + (AXB)^* > CC^* \tag{6.17}$$

if and only if $r(A) = r(B) = m$. In this case, the general solution to (6.17) can be written as

$$X = \frac{1}{2}A^\dagger CC^* B^\dagger + A^\dagger UU^* B^\dagger + [\, I_p, \, 0 \,] F_G V E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} - [\, 0, \, I_p \,] E_H V^* F_G \begin{bmatrix} 0 \\ I_q \end{bmatrix} + W - A^\dagger A W B B^\dagger, \tag{6.18}$$

where $U \in \mathbb{C}^{m\times m}$ is any matrix with $r(U) = m$, and $V \in \mathbb{C}^{(p+q)\times(p+q)}$ and $W \in \mathbb{C}^{p\times q}$ are arbitrary.

(c) *There exists an* X *that satisfies*

$$AXB + (AXB)^* + CC^* \le 0 \tag{6.19}$$

if and only if (6.7) *holds. In this case, the general solution to* (6.19) *can be written as*

$$\begin{aligned} X &= -\frac{1}{2}A^\dagger CC^* B^\dagger - A^\dagger E_M UU^* E_M B^\dagger + [\, I_p,\, 0\,]F_G V E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} \\ &\quad -[\, 0,\, I_p\,]E_H V^* F_G \begin{bmatrix} 0 \\ I_q \end{bmatrix} + W - A^\dagger AWBB^\dagger, \end{aligned} \tag{6.20}$$

where $U \in \mathbb{C}^{m\times m}$, *and* $V \in \mathbb{C}^{(p+q)\times(p+q)}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(d) *There exists an* X *that satisfies the following matrix inequality*

$$AXB + (AXB)^* + CC^* < 0 \tag{6.21}$$

if and only if $r(A) = r(B) = m$. *In this case, the general solution to* (6.21) *can be written as*

$$\begin{aligned} X &= -\frac{1}{2}A^\dagger CC^* B^\dagger - A^\dagger UU^* B^\dagger + [\, I_p,\, 0\,]F_G V E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} \\ &\quad -[\, 0,\, I_p\,]E_H V^* F_G \begin{bmatrix} 0 \\ I_q \end{bmatrix} + W - A^\dagger AWBB^\dagger, \end{aligned} \tag{6.22}$$

where $U \in \mathbb{C}^{m\times m}$ *is any matrix with* $r(U) = m$, *and and* $V \in \mathbb{C}^{(p+q)\times(p+q)}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

Proof. Inequality (6.15) is equivalent to

$$AXB + (AXB)^* = CC^* + YY^* \tag{6.23}$$

for some Y. From Lemma 2.5(b), the equation is solvable for X if and only if

$$E_A YY^* = -E_A CC^*, \quad F_B YY^* = -F_B CC^*, \tag{6.24}$$

this is,

$$\begin{bmatrix} E_A \\ F_B \end{bmatrix} YY^* = -\begin{bmatrix} E_A CC^* \\ F_B CC^* \end{bmatrix}. \tag{6.25}$$

From Lemma 2.5(b), (6.25) is solvable for YY^* if and only if

$$\begin{bmatrix} E_A \\ F_B \end{bmatrix} CC^* [\, E_A,\, F_B\,] \le 0,$$

which is obviously equivalent to $E_ACC^* = F_BCC^* = 0$, i.e., (6.7) holds. In this case, the general solution to (6.25) can be written as

$$\begin{aligned} YY^* &= (I_m - [\, E_A,\, F_B \,][\, E_A,\, F_B \,]^\dagger)UU^*(I_m - [\, E_A,\, F_B \,][\, E_A,\, F_B \,]^\dagger) \\ &= E_MUU^*E_M, \end{aligned}$$

where U is an arbitrary matrix. Substituting the YY^* into (6.23) gives

$$AXB + (AXB)^* = CC^* + E_MUU^*E_M. \tag{6.26}$$

From Lemma 6.1(b), the general solution to (6.26) is

$$\begin{aligned} X \;&=\; \frac{1}{2}A^\dagger CC^*B^\dagger + \frac{1}{2}A^\dagger E_MUU^*E_MB^\dagger + [\, I_p,\, 0\,]F_GVE_H\begin{bmatrix} I_q \\ 0 \end{bmatrix} \\ &\quad -[\, 0,\, I_p\,]E_HV^*F_G\begin{bmatrix} 0 \\ I_q \end{bmatrix} + W - A^\dagger AWBB^\dagger, \end{aligned}$$

establishing (6.16). It can be seen from (6.26) that (6.17) holds if and only if $r(E_MU) = m$. Applying (1.15) gives

$$\begin{aligned} r(E_MU) = m - [\, E_A,\, F_B \,] &= m - r(E_A) - r(AA^\dagger F_B) \\ &= r(A) + r(B) - r\begin{bmatrix} AA^\dagger \\ B \end{bmatrix} \\ &= r(A) + r(B) - r[\, A,\, B^* \,]. \end{aligned}$$

Hence $r(E_MU) = m$ if and only if $r(A) = r(B) = m$. In this case, $E_M = I_m$, and therefore (b) follows from (a). Replacing X with $-X$ in (a) and (b) leads to (c) and (d). □

Corollary 6.3 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times m}$ be given, and denote $M = [\, E_A,\, F_B \,]$, $G = [\, A,\, B^* \,]$ and $H = [\, B^*,\, A\,]^*$. Then,*

(a) *The general solution to*

$$AXB + (AXB)^* \geq 0 \tag{6.27}$$

can be written as

$$\begin{aligned} X \;&=\; A^\dagger E_MUU^*E_MB^\dagger + [\, I_p,\, 0\,]F_GVE_H\begin{bmatrix} I_q \\ 0 \end{bmatrix} \\ &\quad -[\, 0,\, I_p\,]E_HV^*F_G\begin{bmatrix} 0 \\ I_q \end{bmatrix} + W - A^\dagger AWBB^\dagger, \end{aligned} \tag{6.28}$$

where $U \in \mathbb{C}^{m\times m}$, and $V \in \mathbb{C}^{(p+q)\times(p+q)}$ and $W \in \mathbb{C}^{p\times q}$ are arbitrary.

(b) *There exists an* X *that satisfies the following matrix inequality*

$$AXB + (AXB)^* > 0 \tag{6.29}$$

if and only if $r(A) = r(B) = m$*. In this case, the general solution to* (6.29) *can be written as*

$$\begin{aligned} X \;=\;& A^{\dagger}UU^*B^{\dagger} + [\,I_p,\,0\,]F_G V E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} \\ & -[\,0,\,I_p\,]E_H V^* F_G \begin{bmatrix} 0 \\ I_q \end{bmatrix} + W - A^{\dagger}AWBB^{\dagger}, \end{aligned} \tag{6.30}$$

where $U \in \mathbb{C}^{m\times m}$ *is any matrix with* $r(U) = m$*, and* $V \in \mathbb{C}^{(p+q)\times(p+q)}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(c) *The general solution to*

$$AXB + (AXB)^* \leq 0 \tag{6.31}$$

can be written as

$$\begin{aligned} X \;=\;& -A^{\dagger}E_M UU^* E_M B^{\dagger} + \; [\,I_p,\,0\,]F_G V E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} \\ & -[\,0,\,I_p\,]E_H V^* F_G \begin{bmatrix} 0 \\ I_q \end{bmatrix} + W - A^{\dagger}AWBB^{\dagger}, \end{aligned} \tag{6.32}$$

where $U \in \mathbb{C}^{m\times m}$*,* $V \in \mathbb{C}^{(p+q)\times(p+q)}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(d) *There exists an* X *that satisfies the following matrix inequality*

$$AXB + (AXB)^* < 0 \tag{6.33}$$

if and only if $r(A) = r(B) = m$*. In this case, the general solution to* (6.33) *can be written as*

$$\begin{aligned} X \;=\;& -A^{\dagger}UU^*B^{\dagger} + [\,I_p,\,0\,]F_G V E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} \\ & -[\,0,\,I_p\,]E_H V^* F_G \begin{bmatrix} 0 \\ I_q \end{bmatrix} + W - A^{\dagger}AWBB^{\dagger}, \end{aligned} \tag{6.34}$$

where $U \in \mathbb{C}^{m\times m}$ *is any matrix with* $r(U) = m$*, and* $V \in \mathbb{C}^{(p+q)\times(p+q)}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

7 Concluding remarks

We solved four types of matrix inequalities through generalized inverses of matrices, and considered various special cases of the inequalities. The results obtained can be used to characterize structures of unknown matrices satisfying various matrix inequalities in Löwner partial ordering. On the other hand, we believe the work in the paper will motivate further investigation to various general matrix inequalities in Löwner partial ordering, such as,

(a) $AX + YB \geq C$ for $C = C^*$.

(b) $AXA^* + BYB^* \geq C$ for $C = C^*$.

(c) $AXA^* \geq B$ and $CXC^* \geq D$ for $B = B^*$ and $D = D^*$.

(d) $A : X \geq B$, where $A \geq 0$, $B \geq 0$ and $A : X = A(A + X)^{\dagger}X$ is the parallel sum of A and X.

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