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Abstract

This paper presents a simple derivation of eigenvalues, and left and right eigenvectors of a square nonsymmetric matrix with remarkable features. The proof is based on a special Vandermonde matrix. The matrix appears in sampling theory, and its properly normalized right eigenvectors give the inclusion probabilities.

Keywords: Integer spectra, Vandermonde matrix, inverse Vandermonde matrix

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1 Introduction

Recently Nahtman & von Rosen (2010) introduced a new class of matrices with distinguishing algebraic features. The authors were inspired by Bondesson & Traat (2005, 2007) who were the first to discuss how to derive the eigenvalues, and the left and right eigenvectors of certain nonsymmetric matrices with integer spectrum. Their interest in the topic stem from statistical survey sampling problems. For statistical details concerning these problems see Chen & Liu (1997), Airis (1999), Traat et al. (2004) and Bondesson & Traat (2005).

The class of matrices which was introduced in Nahtman & von Rosen (2010) is presented in the next definition. All matrices considered in this paper are real valued, although extension to complex valued matrices is fairly straightforward.

Definition 1 *A square nonsymmetric matrix $B = (b_{ij})$ of order n belongs to the $\mathbb{B}_n$-class if its elements satisfy the following conditions:*

$$\begin{aligned} b_{ii} &= \sum_{\substack{j=1,\\ j\neq i}}^{n} b_{ji}, \qquad i = 1, \dots, n, && (1)\\ b_{ij} + b_{ji} &= 1, \qquad j \neq i,\ i, j = 1, \dots, n, && (2)\\ b_{ij} - b_{ik} &= \frac{b_{ij} b_{ki}}{b_{kj}},\ b_{kj} \neq 0, \qquad j \neq k,\ i \neq k, j;\ i, j, k = 1, \dots, n. && (3) \end{aligned}$$

Instead of (3) one may use $b_{kj} = b_{ij} b_{ki}/(b_{ij} - b_{ik})$ or $b_{ij} b_{kj} = b_{ik} b_{kj} + b_{ij} b_{ki}$. A matrix belonging to $\mathbb{B}_4$ equals

$$B = \begin{pmatrix} b_{21} + b_{31} + b_{41} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{12} + b_{32} + b_{42} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{13} + b_{23} + b_{43} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{14} + b_{24} + b_{34} \end{pmatrix},$$

where $b_{23} = b_{13} b_{21}/(b_{13} - b_{12})$, $b_{24} = b_{14} b_{21}/(b_{14} - b_{12})$ and $b_{34} = b_{14} b_{31}/(b_{14} - b_{13})$. The most interesting property of the $\mathbb{B}_n$-class is that the spectra of the matrices consist of the consecutive integers $\{0, 1, \dots, n-1\}$, i.e. the eigenvalues do not depend on the values of the elements of $B \in \mathbb{B}_n$. Moreover, among others, the sum of the products of the off-diagonal row elements and the sum of the products of the off-diagonal column elements both equal 1, i.e.

$$\sum_{i=1}^{n} \prod_{\substack{j=1\\ j\neq i}}^{n} b_{ij} = 1, \qquad \sum_{j=1}^{n} \prod_{\substack{i=1\\ i\neq j}}^{n} b_{ij} = 1.$$

The left and right eigenvectors were also presented in Nahtman & von Rosen (2010). Furthermore, they showed that for any $B \in \mathbb{B}_n$, $B = P^{-1}TP$, where P is lower triangular and T is an upper triangular matrix. Hence, B is similar to T and the eigenvalues to B can be found on the main diagonal of T. It was also shown that T can be decomposed as $T = D_R V D_L$, where D_R and D_L are diagonal matrices and V is a Vandermonde matrix given by

$$V = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -c_{12} & -c_{13} & \cdots & -c_{1n} \\ 1 & (-c_{12})^2 & (-c_{13})^2 & \cdots & (-c_{1n})^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (-c_{12})^{n-1} & (-c_{13})^{n-1} & \cdots & (-c_{1n})^{n-1} \end{pmatrix}, \tag{4}$$

where

$$c_{ij} = \frac{b_{ji}}{b_{ij}}. \tag{5}$$

The diagonal matrix D_R will be specified in Section 3. Moreover, from Nahtman & von Rosen (2010) it follows that

$$B = D_R^{-1} V^{-1} \Lambda V D_R, \tag{6}$$

where the eigenvalues are collected in the diagonal matrix $\Lambda = (\lambda_{ii})$, $\lambda_{ii} = n-i$. Indeed, it follows from (6) that the left and right eigenvectors equal VD_R and $D_R^{-1}V^{-1}$, respectively. Moreover, $B^k = D_R^{-1}V^{-1}\Lambda^k V D_R$, $k = 1, 2, \ldots, n$.

The starting point for writing this paper is the decomposition in (6) which in a technical and lengthy way was proven in Nahtman & von Rosen (2010). In several places statements were proved via induction and usually one had to consider three different cases. Moreover, symmetry arguments were used which sometimes were not easy to follow. Therefore, the aim of this paper is to present a new proof of (6) which is based on straightforward calculations of $V^{-1}\Lambda V$. This, in turn, leads to a new alternative approach for obtaining eigenvalues and eigenvectors of B given in Definition 1.1.

2 Preparation

Consider the $n \times n$ Vandermonde matrix $W = (v_{ij})$, $v_{ij} = c_i^{j-1}$. It is well known (e.g. see Macon & Spitzbart, 1958; El-Mikkaway, 2003) that

$$W^{-1} = \sum_{i=1}^{n}\sum_{j=1}^{n} \sum_{\substack{i_0<i_1<i_2,\cdots<i_{n-j} \\ i_1\neq i, i_2\neq i,\ldots,i_{n-j}\neq i}}^{[1,n]} \prod_{m=1}^{n-j} c_{i_m}(-1)^{n-j} Q^{-1} e_i e_j', \quad i_0 = 0, \tag{7}$$

where

$$Q = \prod_{\substack{j=1 \\ j \neq i}}^{n} (c_i - c_j), \tag{8}$$

e_i is the ith unit vector, i.e. the ith column of the identity matrix I_n of size $n \times n$ and

$$\sum_{\substack{i_0 < i_1 < i_2, \ldots, < i_{n-j} \\ i_1 \neq i, i_2 \neq i, \cdots < i_{n-j} \neq i}}^{[1,n]} = \sum_{\substack{i_1 = i_0 + 1 \\ i_1 \neq i}}^{j+1} \sum_{\substack{i_2 = i_1 + 1 \\ i_2 \neq i}}^{j+2} \cdots \sum_{\substack{i_{n-j} = i_{n-j-1} + 1 \\ i_{n-j} \neq i}}^{n} .$$

Straightforward calculations can be used to show (7). First, it is noted that

$$\sum_{j=1}^{n} \sum_{\substack{i_0 < i_1 < i_2, \cdots < i_{n-j} \\ i_1 \neq i, i_2 \neq i, \ldots, i_{n-j} \neq i}}^{[1,n]} \prod_{m=1}^{n-j} c_{i_m} (-1)^{n-j} = \prod_{\substack{j=1 \\ j \neq i}}^{n} (1 - c_j), \qquad i_0 = 0, \tag{9}$$

which can be verified by comparing the terms on the left and right sides. If for some k, $c_k = 1$, then both sides in (9) equal 0. Moreover, applying (9) to c_i/a, for any $a \neq 0$, we obtain

$$\sum_{j=1}^{n} \sum_{\substack{i_0 < i_1 < i_2, \cdots < i_{n-j} \\ i_1 \neq i, i_2 \neq i, \ldots, i_{n-j} \neq i}}^{[1,n]} \prod_{m=1}^{n-j} c_{i_m} (-1)^{n-j} a^{j-1} = \prod_{\substack{j=1 \\ j \neq i}}^{n} (a - c_j), \qquad i_0 = 0. \tag{10}$$

If $a = c_i$, from (10) it follows

$$\sum_{j=1}^{n} \sum_{\substack{i_0 < i_1 < i_2, \cdots < i_{n-j} \\ i_1 \neq i, i_2 \neq i, \ldots, i_{n-j} \neq i}}^{[1,n]} \prod_{m=1}^{n-j} c_{i_m} (-1)^{n-j} c_i^{j-1} = \prod_{\substack{j=1 \\ j \neq i}}^{n} (c_i - c_j) = Q, \qquad i_0 = 0,$$

where Q is given by (8), whereas if $a = c_l$ and $c_l = c_k$, for some k, then

$$\sum_{j=1}^{n} \sum_{\substack{i_0 < i_1 < i_2, \cdots < i_{n-j} \\ i_1 \neq i, i_2 \neq i, \ldots, i_{n-j} \neq i}}^{[1,n]} \prod_{m=1}^{n-j} c_{i_m} (-1)^{n-j} c_l^{j-1} = 0, \qquad i_0 = 0.$$

Indeed, the two last relations lead to the expression for the inverse Vandermonde matrix given in (7) since $W^{-1}W = I_n$ holds. However, in this paper we are interested in calculating $W^{-1}\Lambda W$, where $W = V$ and V is given in (4). Therefore, we multiply both sides in (10) by a^{-n+1}, differentiate both sides with respect to a and finally multiply both sides by a^n. Now the following lemma can be formulated.

Lemma 1

$$\sum_{j=1}^{n} \sum_{\substack{i_0<i_1<i_2,\cdots<i_{n-j} \\ i_1\neq i, i_2\neq i,\ldots,i_{n-j}\neq i}}^{[1,n]} \prod_{m=1}^{n-j} c_{i_m} (-1)^{n-j} a^{j-1} (n-j)$$
$$= (n-1)\prod_{\substack{j=1 \\ j\neq i}}^{n} (a-c_j) - a\sum_{l=1}^{n}(a-c_l)^{-1} \prod_{\substack{m=1 \\ m\neq i}}^{n} (a-c_m), \qquad i_0 = 0.$$

Via the lemma the following useful relationships can be obtained: If $a = c_i$,

$$\sum_{j=1}^{n} \sum_{\substack{i_0<i_1<i_2,\cdots<i_{n-j} \\ i_1\neq i, i_2\neq i,\ldots,i_{n-j}\neq i}}^{[1,n]} \prod_{m=1}^{n-j} c_{i_m} (-1)^{n-j} c_i^{j-1} (n-j)$$
$$= (n-1)\prod_{\substack{j=1 \\ j\neq i}}^{n} (c_i-c_j) - c_i\sum_{l=1}^{n}(c_i-c_l)^{-1} \prod_{\substack{m=1 \\ m\neq i}}^{n} (c_i-c_m)$$
$$= ((n-1) - c_i\sum_{l=1}^{n}(c_i-c_l)^{-1})Q, \quad i_0 = 0 \tag{11}$$

and if $c_s = c_k$, for some k,

$$\sum_{j=1}^{n} \sum_{\substack{i_0<i_1<i_2,\cdots<i_{n-j} \\ i_1\neq i, i_2\neq i,\ldots,i_{n-j}\neq i}}^{[1,n]} \prod_{m=1}^{n-j} c_{i_m} (-1)^{n-j} c_i^{j-1} (n-j)$$
$$= -c_s \prod_{\substack{m=1 \\ m\neq i \\ m\neq s}}^{n} (c_s-c_m), \qquad i_0 = 0. \tag{12}$$

3 Main result

In this section we will show how the structure of the matrices in the $\mathbb{B}_n$-class is connected to $V^{-1}\Lambda V$, where V is given in (4). First, two easily proven auxiliary results are displayed.

Lemma 2 *Let $B \in \mathbb{B}_n$ and c_{ij} be given in (5). Then,*

1. $c_{ij}^{-1} = c_{ji}, \quad i \neq j,$
2. $c_{ki}c_{jk} = -c_{ji}, \quad k \neq i,\ j \neq k,\ i \neq j.$

Moreover, the matrix D_R given in the introduction equals

$$D_R = \sum_{i=1}^{n} b_{1i}^{I_{\{i>1\}}} \prod_{\substack{l=2\\ l\neq i}}^{n} b_{li} e_i e_i',$$

where $I_{\{i>1\}}$ is the indicator function, i.e. it equals 0 if $i = 1$ and otherwise its value is 1.

We first consider the diagonal elements in $D_R^{-1}V^{-1}\Lambda V D_R$ which are the same as the diagonal elements of $V^{-1}\Lambda V$. Thus, we have to show that $(V^{-1}\Lambda V)_{ii} = b_{ii}$. Using (11), and remembering that we have to multiply (11) by Q^{-1} in order to consider V^{-1}, yields

$$\begin{aligned}(n-1) - c_i \sum_{\substack{l=1\\ l\neq i}}^{n} (c_i - c_l)^{-1} &= (n-1) - \sum_{\substack{l=1\\ l\neq i}}^{n} (1 - \frac{c_l}{c_i})^{-1} \\ = (n-1) - \sum_{\substack{l=1\\ l\neq i}}^{n} (1 + c_{il})^{-1} &= n - 1 - \sum_{\substack{l=1\\ l\neq i}}^{n} b_{il} = \sum_{\substack{l=1\\ l\neq i}}^{n} b_{li} = b_{ii},\end{aligned}$$

where in the calculations Lemma 3.1 and Definition 1.1 have been used.

Finally the off-diagonal elements of $D_R^{-1}V^{-1}\Lambda V D_R$ are considered. From (12) and the definition of D_R it follows that

$$(D_R^{-1}V^{-1}\Lambda V D_R)_{ir} = -c_r \prod_{\substack{k=1\\ k\neq i\\ k\neq r}}^{n} b_{li}(c_r - c_k) \prod_{\substack{j=1\\ j\neq i}}^{n} (c_i - c_j)^{-1} H, \tag{13}$$

where by definition of D_R

$$H = b_{1r}^{I_{\{r>1\}}} \prod_{\substack{l=2\\ l\neq r}}^{n} b_{lr} \left(b_{1r}^{I_{\{r>1\}}} \prod_{\substack{l=2\\ l\neq r}}^{n} b_{lr} \right)^{-1}. \tag{14}$$

From here a chain of manipulation starts, i.e. (13) is identical to

$$
\begin{aligned}
&-c_r^{n-1}\prod_{\substack{k=1\\k\neq i\\k\neq r}}^{n}(1-\frac{c_k}{c_r})c_i^{-(n-1)}\prod_{\substack{j=1\\j\neq i}}^{n}(1-\frac{c_j}{c_i})^{-1}H\\
&=-c_r^{n-1}\prod_{\substack{k=1\\k\neq i\\k\neq r}}^{n}(1+c_{rk})c_i^{-(n-1)}\prod_{\substack{j=1\\j\neq i}}^{n}(1+c_{ij})^{-1}H\\
&=-c_r^{n-1}\prod_{\substack{k=1\\k\neq i\\k\neq r}}^{n}b_{rk}^{-1}c_i^{-(n-1)}\prod_{\substack{j=1\\j\neq i}}^{n}b_{ij}H\\
&=-c_r^{n-1}\prod_{\substack{k=2\\k\neq i\\k\neq r}}^{n}c_{rk}c_{r1}^{I_{\{r>1\}}}b_{ir}c_i^{-(n-1)}c_{i1}^{I_{\{r>1\}}}\prod_{\substack{l=2\\l\neq i}}^{n}c_{li}\\
&=-c_r^{n-1}\prod_{\substack{k=1\\k\neq i\\k\neq r}}^{n}c_{rk}\prod_{\substack{l=1\\l\neq i}}^{n}c_{li}c_i^{-(n-1)}b_{ir}\\
&=c_{1r}\prod_{\substack{k=1\\k\neq i\\k\neq r}}^{n}c_{1k}\prod_{\substack{l=1\\l\neq i}}^{n}c_{l1}b_{ir}=c_{1r}c_{r1}b_{ir}=b_{ir}.
\end{aligned}
$$

Thus, we have obtained B and simultaneously derived the integer spectra and the structure of the left and right eigenvectors. The proof is much shorter than the one given in Nahtman & von Rosen (2010).

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