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Abstract

In this paper the extended growth curve model is considered. The literature comprises two versions of the model. These models can be connected by one-to-one reparameterizations but since estimators are non-linear it is not obvious how to transmit properties of estimators from one model to another. Since it is only for one of the models where detailed knowledge concerning estimators is available the object in this paper is therefore to present uniqueness properties and moment relations for the estimators of the second model. For comparison reasons properties of the other model are also presented, however, without proofs. It is worth to observe that the presented proofs of uniqueness for linear combinations of estimators is valid for both models and is indeed a simplification of proofs given for one of the models.

Keywords: Extended growth curve model, maximum likelihood estimators, moments, estimability, Kiefer optimality.

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1 Introduction

In experiments, in which more than one characteristic of every treatment is measured, multivariate linear models may be applied. A well known and interesting multivariate linear model is the growth curve model (GCM) due to [8] which belongs to the curved exponential family. Many results and references can be found in [4, Chapter 4]. Among others explicit maximum likelihood estimators (MLE), estimability conditions, moments and approximative distributions of the estimators are available.

Markiewicz and Szczepańska [7] considered the GCM with additional nuisance parameters. They determined estimators of the parameters of interest as well as presented the first and second order moments of this estimator. Kiefer optimal designs and relations between optimality under univariate and multivariate models were given. Moreover, the GCM with two nuisance parameters was considered in [3]. The authors gave estimators of the parameters and obtained appropriate moment relations to determine Kiefer optimal designs.

Consider a linear model

$$\mathbf{y} = \mathbf{A}_1\boldsymbol{\beta}_1 + \mathbf{A}_2\boldsymbol{\beta}_2 + \mathbf{A}_3\boldsymbol{\beta}_3 + \boldsymbol{\epsilon}, \tag{1.1}$$

where $\mathbf{A}_i \in \mathbb{R}^{n\times m_i}$, $i = 1,2,3$, are known design matrices and $\mathbf{y} \in \mathbb{R}^n$ is an observable random vector, which depends linearly on several parameters. The model in (1.1) represents measurements on a single response variable y. Here $\boldsymbol{\beta}_i \in \mathbb{R}^{m_i}$, $i = 1,2,3$, are vectors of parameters, and $\boldsymbol{\epsilon} \in \mathbb{R}^n$ is a vector of normally distributed random errors with expectation $E[\boldsymbol{\epsilon}] = \mathbf{0}$, and dispersion matrix $D[\boldsymbol{\epsilon}] = \mathbf{I}_n$, where $\mathbf{I}_n$ is the identity matrix of size $n \times n$.

If we are measuring p response variables on each sampling unit we can extend (1.1) and consider the following multivariate linear model

$$\mathbf{Y} = \mathbf{A}_1\mathbf{B}_1\mathbf{C}_1 + \mathbf{A}_2\mathbf{B}_2\mathbf{C}_2 + \mathbf{A}_3\mathbf{B}_3\mathbf{C}_3 + \mathbf{E}, \tag{1.2}$$

where in addition to $\mathbf{A}_i$ the matrices $\mathbf{C}_i \in \mathbb{R}^{q_i\times p}$, $i = 1,2,3$, are known. The matrix $\mathbf{Y} \in \mathbb{R}^{n\times p}$ is an observations matrix and $\mathbf{B}_i \in \mathbb{R}^{m_i\times q_i}$, $i = 1,2,3$, are matrices of unknown parameters. The matrix $\mathbf{E} \in \mathbb{R}^{n\times p}$ is a matrix of random errors, normally distributed, with expectation $E[\mathbf{E}] = \mathbf{0}$ and with dispersion matrix $D[\mathbf{E}] = D[\mathrm{vec}(\mathbf{E})] = \boldsymbol{\Sigma} \otimes \mathbf{I}_n$, where $\boldsymbol{\Sigma} \in \mathbb{R}_p^{>}$ is an unknown positive definite matrix, $\mathrm{vec}(\mathbf{E})$ denotes the vector in $\mathbb{R}^{pn}$ formed by putting the columns of $\mathbf{E} \in \mathbb{R}^{n\times p}$ under each other, starting from the left, and $\otimes$ denotes the Kronecker product. The matrices $\mathbf{A}_i$ (between individuals design matrices) are used to design the experiment, i.e. lay out treatments in an appropriate way, whereas the $\mathbf{C}_i$ matrices (within individuals design matrices) are used to describe the relation between the response variables.

The model in (1.2) will be called extended growth curve model (EGCM). As seen it is a generalized version of the GCM and is sometimes termed sums of profiles model (see [9]). The model may be viewed as a multivariate seemingly unrelated regression (SUR) model. However to obtain explicit maximum likelihood estimators a nested subspace condition has to be imposed. This can be performed in two different ways leading to different parameterizations. However, it is only for one of them where a lot of detailed knowledge such as uniqueness conditions for MLEs, moments and asymptotics has been presented (e.g. [4, Chapter 4]: observe that the role of $\mathbf{A}_i$ and $\mathbf{C}_i$ in this work are interchanged with the role of the same matrices in the present paper). When discussing Kiefer optimality, unfortunately, we need results for the estimators of parameters in the other parametrization.

In the subsequent we are going to refer to the two models as Model I and Model II: ($\mathcal{R}(\bullet)$ denotes the column space)

Definition 1.1. *Let all matrices be the same as in (1.2).*

Model I:

$$\mathbf{Y} = \mathbf{A}_1\mathbf{B}_1\mathbf{C}_1 + \mathbf{A}_2\mathbf{B}_2\mathbf{C}_2 + \mathbf{A}_3\mathbf{B}_3\mathbf{C}_3 + \mathbf{E}, \qquad \mathcal{R}(\mathbf{C}_3') \subseteq \mathcal{R}(\mathbf{C}_2') \subseteq \mathcal{R}(\mathbf{C}_1'); \tag{1.3}$$

Model II:

$$\mathbf{Y} = \mathbf{A}_1\mathbf{B}_1\mathbf{C}_1 + \mathbf{A}_2\mathbf{B}_2\mathbf{C}_2 + \mathbf{A}_3\mathbf{B}_3\mathbf{C}_3 + \mathbf{E}, \qquad \mathcal{R}(\mathbf{A}_3) \subseteq \mathcal{R}(\mathbf{A}_2) \subseteq \mathcal{R}(\mathbf{A}_1). \tag{1.4}$$

It will now be shown that Model I and Model II indeed are equivalent, i.e. via reparameterizations one can derive Model I from Model II or vice versa. From (1.3) it follows that there exist matrices $\mathbf{H}_1$ and $\mathbf{H}_2$ such that

$$\mathcal{R}(\mathbf{C}_1') = \mathcal{R}(\mathbf{C}_2') \boxplus \mathcal{R}(\mathbf{H}_1'),$$
$$\mathcal{R}(\mathbf{C}_2') = \mathcal{R}(\mathbf{C}_3') \boxplus \mathcal{R}(\mathbf{H}_2'),$$

where $\boxplus$ denotes the orthogonal sum of subspaces. Let $\mathbf{\Theta}_1 = (\mathbf{\Theta}_{11} : \mathbf{\Theta}_{12} : \mathbf{\Theta}_{13})$ and $\mathbf{\Theta}_2 = (\mathbf{\Theta}_{21} : \mathbf{\Theta}_{22})$. Then, Model I is equivalent to

$$\mathbf{Y} = (\mathbf{A}_1 : \mathbf{A}_2 : \mathbf{A}_3)(\mathbf{\Theta}_{11}' : \mathbf{\Theta}_{21}' : \mathbf{B}_3')'\mathbf{C}_3 + (\mathbf{A}_1 : \mathbf{A}_2)(\mathbf{\Theta}_{12}' : \mathbf{\Theta}_{22}')'\mathbf{H}_2 + \mathbf{A}_1\mathbf{\Theta}_{13}\mathbf{H}_1 + \mathbf{E}$$

which according to Definition 1.1 is of Model II type. The main problem is that because of non-linearity of estimators it is not obvious how to transmit properties of estimators from one model to the other, in particular moment relations. This will become clear when explicit estimators are presented.

In this paper we are mainly interested in the EGCM model with the range condition on the within-individuals design matrices, i.e. Model I. The reason for this is that in models connected to experimental designs the ranges of between-individuals design matrices should be disjoint or, the intersection should be as small as possible (cf. regression models, interference models). For example, in block designs, the common space of these matrices is the vector of ones.

The aim of this paper is to present results in parallel for both Model I and Model II and in particular derive new results for Model I. Estimators of the unknown parameters are presented as well as moments of these estimators. Conditions for uniqueness of the estimators will also be given.

Throughout the paper we use the following notation. Let $\mathbf{P}_X = \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'$ and $\mathbf{Q}_X = \mathbf{I}_m - \mathbf{P}_X$ denote the orthogonal projectors on $\mathcal{R}(\mathbf{X})$ and its orthocomplement, respectively. Moreover, $\mathbf{X}^-$ denotes an arbitrary generalized inverse of the matrix $\mathbf{X}$ and $\mathbf{X}^o$ is any matrix spanning $\mathcal{R}(\mathbf{X})^\perp$. For a positive definite $\mathbf{B}$ we denote $\mathbf{P}_{X;B} = \mathbf{X}(\mathbf{X}'\mathbf{B}\mathbf{X})^-\mathbf{X}'\mathbf{B}$ and $\mathbf{Q}_{X;B} = \mathbf{I} - \mathbf{P}_{X;B}$. It follows that $\mathbf{I} = \mathbf{P}_{X;B} + \mathbf{P}'_{X^o;B^{-1}}$ which is equivalent to

$$\mathbf{B} = \mathbf{B}\mathbf{X}(\mathbf{X}'\mathbf{B}\mathbf{X})^-\mathbf{X}'\mathbf{B} + \mathbf{X}^o(\mathbf{X}^{o'}\mathbf{B}^{-1}\mathbf{X})^-\mathbf{X}^{o'} \tag{1.5}$$

a well known formula which often will be utilized in this paper. We use $\text{rank}(\mathbf{X})$ and $\text{tr}\{\mathbf{X}\}$ to denote the rank and the trace of $\mathbf{X}$, respectively. Moreover, sometimes it is written $(\mathbf{A})()'$ instead of $(\mathbf{A})(\mathbf{A})'$.

The Introduction is ended by presenting three examples which illustrate Model I and Model II.

Example 1 - Interference model

Consider an agricultural experiment. Suppose it is desired to compare the yield of v different varieties of grain (treatments). It is likely that there is an interaction between the environment (type of soil, rainfall, drainage, etc.) and the variety of grain which will alter the yields. So, b blocks [sets of experimental plots (units)] are chosen in which the environment is fairly consistent throughout the block; R. A. Fisher and F. Yates, early 1930's.

Let n experimental units (plots) because of extraneous variability be divided into b blocks each of size k where the blocks consist of homogeneous units for. Let v treatments be applied to the units so that each unit receives one treatment. The treatment which is applied to unit j in block i is determined by the design d. In each block the effect of the treatments applied to each unit is measured by a random variable $\mathbf{y}$.

Assume, the response on a given plot may be affected by treatments on neighboring plots as well as by the treatment applied to that plot. Consider experiments with a one-dimensional arrangement of plots in each block, and for which the treatments have different left and right neighbor interference effects. In the literature circular experiments ([2]) and experiments without border plots ([6]) have studied.

A linear model associated with a design d has the form

$$\mathbf{y} = \mathbf{A}_{1,d}\boldsymbol{\beta}_1 + \mathbf{A}_{2,d}\boldsymbol{\beta}_2 + \mathbf{A}_3\boldsymbol{\beta}_3 + \boldsymbol{\epsilon},$$

where $\boldsymbol{\beta}_i$, $i = 1, 2, 3$, are the unknown vectors of treatment effects, neighbor effects, and block effects, respectively, and ϵ is the vector of random errors. The matrix $\mathbf{A}_{1,d} \in \mathbb{R}^{n\times v}$ depends on the design and it is a binary matrix which satisfies $\mathbf{A}_{1,d}\mathbf{1}_v = \mathbf{1}_n$. The matrix $\mathbf{A}_{2,d} = ((\mathbf{I}_b \otimes \mathbf{H})\mathbf{A}_{1,d} : (\mathbf{I}_b \otimes \mathbf{H}')\mathbf{A}_{1,d})$, is a known matrix of neighbor effects, where

$$\mathbf{H} = \begin{pmatrix} \mathbf{0}'_{k-1} & 1 \\ \mathbf{I}_{k-1} & \mathbf{0}_{k-1} \end{pmatrix} \quad \text{or} \quad \mathbf{H} = \begin{pmatrix} \mathbf{0}'_{k-1} & 0 \\ \mathbf{I}_{k-1} & \mathbf{0}_{k-1} \end{pmatrix}$$

for the circular design and for the design without border plots, respectively ($\mathbf{0}_{k-1}$ is a $k-1$ dimensional vector of zeros). The matrix $\mathbf{A}_3 = \mathbf{I}_b \otimes \mathbf{1}_k$ is the design matrix of block effects.

In the literature such a model is called *an interference model with neighbor effects.*

Assume, we measure p characteristics of every treatment. Then, we have the following extension of the interference model:

$$\mathbf{Y} = \mathbf{A}_{1,d}\mathbf{B}_1\mathbf{C}_1 + \mathbf{A}_{2,d}\mathbf{B}_2\mathbf{C}_2 + \mathbf{A}_3\mathbf{B}_3\mathbf{C}_3 + \mathbf{E},$$

where $\mathbf{Y} \in \mathbb{R}^{n\times p}$ is the matrix of observations, $\mathbf{B}_i$, $i = 1, 2, 3$, are the unknown matrices of treatment, neighbor and block effects, respectively, and $\mathbf{C}_i$, $i = 1, 2, 3$, are the restriction matrices.

Assume now, that in the experiment there is no left- and right-neighbor effect and no block effect for the last characteristic, and for the second last characteristic there is no block effect. Then, $\mathbf{C}_1 = \mathbf{I}_p$, $\mathbf{C}_2 = (\mathbf{I}_{p-1}, \mathbf{0}_{p-1})$ and $\mathbf{C}_3 = (\mathbf{I}_{p-2}, \mathbf{0}_{p-2}, \mathbf{0}_{p-2})$ and we obtain Model 1.

Example 2 - Standard cross-over model with carry-over effects

In a setting of repeated measurements design each of a set of n experimental units is, in each of p periods, exposed to one of v treatments. At each period we measure the effect of the treatments applied to each unit by a random variable $\mathbf{y}$. It is assumed that each measurement is influenced by an additive first order residual effect of the treatment to which the unit under consideration has been exposed in the period before. Let consider designs with no residual effects on the first period.

The cross-over model associated with the repeated measurements design is of the form

$$\mathbf{y} = \mathbf{A}_{1,d}\boldsymbol{\beta}_1 + (\mathbf{I}_b \otimes \mathbf{H})\mathbf{A}_{1,d}\boldsymbol{\beta}_2 + (\mathbf{1}_b \otimes \mathbf{I}_k : \mathbf{I}_b \otimes \mathbf{1}_k)\boldsymbol{\beta}_3 + \epsilon$$

with

$$\mathbf{H} = \begin{pmatrix} \mathbf{0}' & 0 \\ \mathbf{I}_{k-1} & \mathbf{0} \end{pmatrix},$$

where $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ are the vectors of treatment and residual effects, and $\boldsymbol{\beta}_3 = (\boldsymbol{\alpha}', \boldsymbol{\beta}')'$ consists of a vector of period effects and a vector of unit effects (Kunert, 1983).

The multivariate extension of the cross-over model is of the form

$$\mathbf{Y} = \mathbf{A}_{1,d}\mathbf{B}_1\mathbf{C}_1 + (\mathbf{I}_b \otimes \mathbf{H})\mathbf{A}_{1,d}\mathbf{B}_2\mathbf{C}_2 + (\mathbf{1}_b \otimes \mathbf{I}_k : \mathbf{I}_b \otimes \mathbf{1}_k)\mathbf{B}_3\mathbf{C}_3 + \mathbf{E},$$

where $\mathbf{Y} \in \mathbb{R}^{n\times p}$ is the matrix of observations, $\mathbf{B}_i$, $i = 1, 2, 3$, are the unknown matrices of treatment, residual and period-unit effects, respectively, while $\mathbf{C}_i$, $i = 1, 2, 3$, are the restriction matrices. If we assume now, that in the experiment there is no period effect and no block effect for the first and second characteristic, and that there is no residual effect for the last characteristic, then $\mathbf{C}_1 = \mathbf{I}_p$, $\mathbf{C}_2 = (\mathbf{I}_{p-1}, \mathbf{0}_{p-1})$ and $\mathbf{C}_3 = (\mathbf{0}_{p-2}, \mathbf{0}_{p-2}, \mathbf{I}_{p-2})$ and we obtain Model 1.

Example 3 - Growth curves

Suppose that we have a random vector $\mathbf{y}$ associated to observations which follows the model

$$\mathbf{y}' = \boldsymbol{\mu} + \boldsymbol{\epsilon}',$$

where $\boldsymbol{\epsilon} \sim N_p(\mathbf{0}, \boldsymbol{\Sigma})$. Suppose that there exist a linear relation among the components in $\boldsymbol{\mu}$, i.e. $\boldsymbol{\mu}' \in \mathcal{R}(\mathbf{C}')$. Thus, $\boldsymbol{\mu} = \boldsymbol{\beta}\mathbf{C}$ for some $\boldsymbol{\beta}$ and $\mathbf{y}' = \boldsymbol{\beta}\mathbf{C} + \boldsymbol{\epsilon}'$. Now suppose that we have n independent observations which all have the same within individual model $\boldsymbol{\mu}' \in \mathcal{R}(\mathbf{C}')$ and that there is a linear model between the independent observation. For example, there are three groups of individuals; one corresponding to a placebo treatment and the others corresponding to two different treatments, respectively. Thus we end up in the following model

$$\mathbf{Y} = \mathbf{ABC} + \mathbf{E},$$

where $\mathbf{Y}' = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)$, $\mathbf{B} = (\boldsymbol{\beta}_1', \boldsymbol{\beta}_2', \boldsymbol{\beta}_3')'$, $\mathbf{E} \sim N_{n,p}(\mathbf{0}, \mathbf{I}, \boldsymbol{\Sigma})$ and

$$\mathbf{A}' = \begin{pmatrix} 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 \end{pmatrix}.$$

Moreover suppose that we have a polynomial growth. Then, for example,

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ t_1 & t_2 & \ldots & t_p \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{q-1} & t_1^{q-1} & \ldots & t_p^{q-1} \end{pmatrix}.$$

In this model all individuals follow the same polynomial growth model. However, if each treatment group follows a polynomial of different order we may for example have the following model

$$\mathbf{Y} = \mathbf{A}_1\mathbf{B}_1\mathbf{C}_1 + \mathbf{A}_2\mathbf{B}_2\mathbf{C}_2 + \mathbf{A}_3\mathbf{B}_3\mathbf{C}_3 + \mathbf{E},$$

where

$$\mathbf{A}_1' = \begin{pmatrix} 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 \end{pmatrix},$$

$$\mathbf{A}_2' = \begin{pmatrix} 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

$$\mathbf{A}_3' = \begin{pmatrix} 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

$$\mathbf{C}_1 = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ t_1 & t_2 & \ldots & t_p \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{q-3} & t_1^{q-3} & \ldots & t_p^{q-3} \end{pmatrix},$$

$$\mathbf{C}_2 = \begin{pmatrix} t_1^{q-2} & t_1^{q-2} & \dots & t_p^{q-2} \end{pmatrix},$$

$$\mathbf{C}_3 = \begin{pmatrix} t_1^{q-1} & t_1^{q-1} & \dots & t_p^{q-1} \end{pmatrix}.$$

Observe that $\mathcal{R}(\mathbf{A}_3) \subseteq \mathcal{R}(\mathbf{A}_2) \subseteq \mathcal{R}(\mathbf{A}_1)$ and thus we have a model which is formulated as Model II. The above example means, for example, that the mean of the placebo group and the treatment groups respectively equal

$$\begin{aligned}
&\beta_{11} + \beta_{12}t + \dots + \beta_{1(q-2)}t^{q-3},\\
&\beta_{21} + \beta_{22}t + \dots + \beta_{2(q-2)}t^{q-3} + \beta_{2(q-1)}t^{q-2},\\
&\beta_{31} + \beta_{32}t + \dots + \beta_{3(q-2)}t^{q-3} + \beta_{3(q-1)}t^{q-2} + \beta_{3q}t^{q-1}.
\end{aligned}$$

2 Maximum likelihood estimators

Maximum likelihood estimators have been presented for Model I as well as Model II.

Theorem 2.1 ([3])**.** *In Model I the maximum likelihood estimators of the parameters equal*

$$\begin{aligned}
\widehat{\mathbf{B}}_1 &= (\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{A}_1'(\mathbf{Y} - \mathbf{A}_2\widehat{\mathbf{B}}_2\mathbf{C}_2 - \mathbf{A}_3\widehat{\mathbf{B}}_3\mathbf{C}_3)\mathbf{S}_3^{-1}\mathbf{C}_1'(\mathbf{C}_1\mathbf{S}_3^{-1}\mathbf{C}_1')^-\\
&\quad +(\mathbf{A}_1')^o\mathbf{Z}_{11}\mathbf{C}_1' + \mathbf{A}_1'\mathbf{Z}_{12}\mathbf{C}_1^{o'},\\
\widehat{\mathbf{B}}_2 &= (\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1}(\mathbf{Y} - \mathbf{A}_3\widehat{\mathbf{B}}_3\mathbf{C}_3)\mathbf{S}_2^{-1}\mathbf{C}_2'(\mathbf{C}_2\mathbf{S}_2^{-1}\mathbf{C}_2')^-\\
&\quad +(\mathbf{A}_2'\mathbf{Q}_{A_1})^o\mathbf{Z}_{21}\mathbf{C}_2' + \mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{Z}_{22}\mathbf{C}_2^{o'},\\
\widehat{\mathbf{B}}_3 &= (\mathbf{A}_3'\mathbf{Q}_{(A_1:A_2)}\mathbf{A}_3)^-\mathbf{A}_3'\mathbf{Q}_{(A_1:A_2)}\mathbf{Y}\mathbf{S}_1^{-1}\mathbf{C}_3'(\mathbf{C}_3\mathbf{S}_1^{-1}\mathbf{C}_3')^-\\
&\quad +(\mathbf{A}_3'\mathbf{Q}_{(A_1:A_2)})^o\mathbf{Z}_{31}\mathbf{C}_3' + \mathbf{A}_3'\mathbf{Q}_{(A_1:A_2)}\mathbf{Z}_{32}\mathbf{C}_3^{o'},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{S}_1 &= \mathbf{Y}'\mathbf{Q}_{(A_1:A_2:A_3)}\mathbf{Y}, \quad \mathbf{S}_2 = \mathbf{S}_1 + \mathbf{Q}_{C_3';S_1^{-1}}\mathbf{Y}'\mathbf{P}_{Q_{(A_1:A_2)}A_3}\mathbf{Y}\mathbf{Q}'_{C_3';S_1^{-1}},\\
\mathbf{S}_3 &= \mathbf{S}_2 + \mathbf{Q}_{C_2';S_2^{-1}}\mathbf{Y}'\mathbf{P}_{Q_{A_1}A_2}\mathbf{Y}\mathbf{Q}'_{C_2';S_2^{-1}},
\end{aligned}$$

and $\mathbf{Z}_{ij}$*,* $i = 1, 2, 3$*,* $j = 1, 2$*, are arbitrary matrices. The ML-estimator of the dispersion matrix can be written*

$$\begin{aligned}
n\widehat{\mathbf{\Sigma}} &= (\mathbf{Y} - \mathbf{A}_1\widehat{\mathbf{B}}_1\mathbf{C}_1 - \mathbf{A}_2\widehat{\mathbf{B}}_2\mathbf{C}_2 - \mathbf{A}_3\widehat{\mathbf{B}}_3\mathbf{C}_3)'()\\
&= \mathbf{S}_3 + \mathbf{Q}_{C_1';S_3^{-1}}\mathbf{Y}'\mathbf{P}_{A_1}\mathbf{Y}\mathbf{Q}'_{C_1';S_3^{-1}}.
\end{aligned}$$

Theorem 2.2 ([4])**.** *In Model II the maximum likelihood estimators of the parameters equal*

$$\begin{aligned}
\widehat{\mathbf{B}}_1 &= (\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{A}_1'(\mathbf{Y} - \mathbf{A}_2\widehat{\mathbf{B}}_2\mathbf{C}_2 - \mathbf{A}_3\widehat{\mathbf{B}}_3\mathbf{C}_3)\mathbf{S}_1^{-1}\mathbf{C}_1'(\mathbf{C}_1\mathbf{S}_1^{-1}\mathbf{C}_1')^-\\
&\quad +\mathbf{Z}_{11}\mathbf{C}_1^{o'} + \mathbf{A}_1'^{o}\mathbf{Z}_{12}\mathbf{C}_1',\\
\widehat{\mathbf{B}}_2 &= (\mathbf{A}_2'\mathbf{A}_2)^-\mathbf{A}_2'(\mathbf{Y} - \mathbf{A}_3\widehat{\mathbf{B}}_3\mathbf{C}_3)\mathbf{S}_2^{-1}\mathbf{Q}_{C_1';S_1^{-1}}\mathbf{C}_2'(\mathbf{C}_2\mathbf{Q}'_{C_1';S_1^{-1}}\mathbf{S}_2^{-1}\mathbf{Q}_{C_1';S_1^{-1}}\mathbf{C}_2')^-\\
&\quad +\mathbf{Z}_{21}(\mathbf{C}_2\mathbf{Q}'_{C_1';S_1^{-1}})^{o'} + \mathbf{A}_2'^{o}\mathbf{Z}_{22}\mathbf{Q}_{C_1';S_1^{-1}}\mathbf{C}_2',\\
\widehat{\mathbf{B}}_3 &= (\mathbf{A}_3'\mathbf{A}_3)^-\mathbf{A}_3'\mathbf{Y}\mathbf{S}_3^{-1}\mathbf{P}_3\mathbf{C}_3'(\mathbf{C}_3\mathbf{P}_3'\mathbf{S}_3^{-1}\mathbf{P}_3\mathbf{C}_3')^-\\
&\quad +\mathbf{Z}_{31}(\mathbf{C}_3\mathbf{P}_3')^{o'} + \mathbf{A}_3'^{o}\mathbf{Z}_{32}\mathbf{P}_3\mathbf{C}_3',
\end{aligned}$$

where

$$\begin{aligned}\mathbf{S}_1 &= \mathbf{Y}'\mathbf{Q}_{A_1}\mathbf{Y}, \quad \mathbf{S}_2 = \mathbf{S}_1 + \mathbf{Q}_{C_1';S_1^{-1}}\mathbf{Y}'\mathbf{P}_{A_1}\mathbf{Q}_{A_2}\mathbf{P}_{A_1}\mathbf{Y}\mathbf{Q}'_{C_1';S_1^{-1}},\\ \mathbf{S}_3 &= \mathbf{S}_2 + \mathbf{P}_3\mathbf{Y}'\mathbf{P}_{A_1}\mathbf{Q}_{A_3}\mathbf{P}_{A_1}\mathbf{Y}\mathbf{P}'_3,\end{aligned}$$

and $\mathbf{Z}_{ij}$*,* $i = 1,2,3,\ j = 1,2$ *are arbitrary matrices, and*

$$\begin{aligned}n\widehat{\mathbf{\Sigma}} &= (\mathbf{Y} - \mathbf{A}_1\widehat{\mathbf{B}}_1\mathbf{C}_1 - \mathbf{A}_2\widehat{\mathbf{B}}_2\mathbf{C}_2 - \mathbf{A}_3\widehat{\mathbf{B}}_3\mathbf{C}_3)'()\\ &= \mathbf{S}_3 + \mathbf{P}_4\mathbf{Y}'\mathbf{P}_{A_3}\mathbf{Y}\mathbf{P}'_4,\end{aligned}$$

where

$$\begin{aligned}&\mathbf{P}_i = \mathbf{U}_{i-1}\cdot\ldots\cdot\mathbf{U}_1, && i = 3,4,\\ &\mathbf{U}_j = \mathbf{Q}_{P_jC_j';S_j^{-1}}, && j = 1,2,3.\end{aligned}$$

Both Theorem 2.1 and Theorem 2.2 can be obtained by solving the following likelihood equations:

$$\mathbf{0} = \mathbf{A}'_1(\mathbf{Y} - \mathbf{A}_1\mathbf{B}_1\mathbf{C}_1 - \mathbf{A}_2\mathbf{B}_2\mathbf{C}_2 - \mathbf{A}_3\mathbf{B}_3\mathbf{C}_3)\mathbf{\Sigma}^{-1}\mathbf{C}'_1, \tag{2.6}$$

$$\mathbf{0} = \mathbf{A}'_2(\mathbf{Y} - \mathbf{A}_1\mathbf{B}_1\mathbf{C}_1 - \mathbf{A}_2\mathbf{B}_2\mathbf{C}_2 - \mathbf{A}_3\mathbf{B}_3\mathbf{C}_3)\mathbf{\Sigma}^{-1}\mathbf{C}'_2, \tag{2.7}$$

$$\mathbf{0} = \mathbf{A}'_3(\mathbf{Y} - \mathbf{A}_1\mathbf{B}_1\mathbf{C}_1 - \mathbf{A}_2\mathbf{B}_2\mathbf{C}_2 - \mathbf{A}_3\mathbf{B}_3\mathbf{C}_3)\mathbf{\Sigma}^{-1}\mathbf{C}'_3, \tag{2.8}$$

$$n\mathbf{\Sigma} = (\mathbf{Y} - \mathbf{A}_1\mathbf{B}_1\mathbf{C}_1 - \mathbf{A}_2\mathbf{B}_2\mathbf{C}_2 - \mathbf{A}_3\mathbf{B}_3\mathbf{C}_3)'().$$

3 Uniqueness conditions for the MLEs

Consider the Gauss-Markov model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \varepsilon, \quad E(\boldsymbol{\epsilon}) = \mathbf{0}, \quad Cov(\boldsymbol{\epsilon}) = \sigma^2\mathbf{\Sigma},$$

where $\mathbf{\Sigma}$ is known. It is well known that the least squares estimator of a linear function of the parameter vector $\boldsymbol{\beta}$, say $\mathbf{p}'\widehat{\boldsymbol{\beta}}$, is unique if and only if $\mathbf{p}'\boldsymbol{\beta}$ is estimable. The estimability condition may be expressed as

$$\mathbf{p} \in \mathcal{R}(\mathbf{X}').$$

Now observe, that the condition of estimability of $\mathbf{p}'\boldsymbol{\beta}$ under a linear model with nuisance parameters,

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\epsilon},$$

may be expressed as (for more details see e.g. [1])

$$\mathbf{p} \in \mathcal{R}(\mathbf{X}'\mathbf{Q}_Z).$$

Moreover, it is well known that the necessary and sufficient condition of estimability of linear parametric functions $\mathbf{K\Xi L}$ in a multivariate linear model

$$\mathbf{Y} = \mathbf{A\Xi P} + \mathbf{E}, \quad Cov(\mathbf{E}) = \mathbf{\Sigma}\otimes\mathbf{I},$$

where $\mathbf{\Sigma}$ is a known, positive definite matrix, has the form

$$\mathcal{R}(\mathbf{K}') \subset \mathcal{R}(\mathbf{A}') \quad \text{and} \quad \mathcal{R}(\mathbf{L}) \subset \mathcal{R}(\mathbf{P}).$$

This form of the above condition may be obtained by using the "vec" operator and then consider linear spaces generated by Kronecker products, which indeed are tensor spaces. Furthermore, it can be seen that under the multivariate model with nuisance parameters,

$$\mathbf{Y} = \mathbf{A}_1\mathbf{\Xi}\mathbf{P}_1 + \mathbf{A}_2\mathbf{\Theta}\mathbf{P}_2 + \mathbf{E},$$

using the elimination of nuisance parameters, the estimablity condition for $\mathbf{K\Xi L}$ can be written

$$\mathcal{R}(\mathbf{L}' \otimes \mathbf{K}) \subset \mathcal{R}((\mathbf{P}_1 \otimes \mathbf{A}_1')\mathbf{Q}_{P_2' \otimes A_2});$$

for more details see e.g. [3]. These conditions are equivalent to the uniqueness condition of the least squares estimator of $\mathbf{\Xi}$.

Let us consider Models I and II. We are interested in estimation of linear functions of $\mathbf{B}_i$, $i = 1, 2, 3$, which can be presented as $\sum_j \mathbf{K}_j\mathbf{B}_i\mathbf{L}_j$. Estimability conditions for the linear functions of $\mathbf{B}_2$ and $\mathbf{B}_3$ have been presented in [3].

Theorem 3.1. *The linear functions* $\sum_j \mathbf{K}_j\mathbf{B}_i\mathbf{L}_j$, $i = 1, 2, 3$, *are estimable in Model I if and only if*

(i) $\mathcal{R}\left(\sum_j \mathbf{L}_j \otimes \mathbf{K}_j'\right) \subseteq \mathcal{R}\left(\mathbf{C}_1\mathbf{Q}_{C_2'} \otimes \mathbf{A}_1'\right) + \mathcal{R}\left(\mathbf{C}_1\mathbf{P}_{C_2'}\mathbf{Q}_{C_3'} \otimes \mathbf{A}_1'\mathbf{Q}_{A_2}\right) + \mathcal{R}\left(\mathbf{C}_1\mathbf{P}_{C_3'} \otimes \mathbf{A}_1'\mathbf{Q}_{(A_2:A_3)}\right)$, *for* $i = 1$,

(ii) $\mathcal{R}\left(\sum_j \mathbf{L}_j \otimes \mathbf{K}_j'\right) \subseteq \mathcal{R}\left(\mathbf{C}_2\mathbf{Q}_{C_3'} \otimes \mathbf{A}_2'\mathbf{Q}_{A_1}\right) + \mathcal{R}\left(\mathbf{C}_2\mathbf{P}_{C_3'} \otimes \mathbf{A}_2'\mathbf{Q}_{(A_1:A_3)}\right)$, *for* $i = 2$,

(iii) $\mathcal{R}\left(\sum_j \mathbf{L}_j \otimes \mathbf{K}_j'\right) \subseteq \mathcal{R}\left(\mathbf{C}_3 \otimes \mathbf{A}_3'\mathbf{Q}_{(A_1:A_2)}\right)$, *for* $i = 3$.

Proof. The bearing idea of the proof is the following. If $\mathbf{\Sigma}$ is known we have a usual Gauss-Markov model. In this case all estimators satisfy, for given $\mathbf{\Sigma}$, (2.6), (2.7) and (2.8). However, it will appear that the uniqueness conditions depend only on the design matrices $\mathbf{A}_i$ and $\mathbf{C}_i$ and are completely unrelated with $\mathbf{\Sigma}$. Thus, for all values of $\mathbf{\Sigma}$, including the MLE, the same conditions for uniqueness are obtained. Hence, we have the complete solution to uniqueness/estimation problems for the EGCM and it suffices to consider models with known $\mathbf{\Sigma}$. Moreover, it is noted that we immediately obtain conditions for both Model I and Model II and $\widehat{\mathbf{\Sigma}}$ in both models is always uniquely estimated.

Now we consider Model I in some detail.

Let $i = 3$. Then, using the "vec" operator and by elimination of nuisance parameters (first $\mathbf{B}_1$ and then $\mathbf{B}_2$), we obtain the following estimability condition:

$$\mathcal{R}\left(\sum_j \mathbf{L}_j \otimes \mathbf{K}_j'\right) \subseteq \mathcal{R}\left((\mathbf{C}_3 \otimes \mathbf{A}_3')\mathbf{Q}_{C_1' \otimes A_1}\mathbf{Q}_{Q_{C_1' \otimes A_1}(C_2' \otimes A_2)}\right).$$

Since $\mathcal{R}(\mathbf{C}_3') \subseteq \mathcal{R}(\mathbf{C}_2') \subseteq \mathcal{R}(\mathbf{C}_1')$ we have

$$(\mathbf{C}_3 \otimes \mathbf{A}_3')\mathbf{Q}_{C_1' \otimes A_1} = \mathbf{C}_3 \otimes \mathbf{A}_3'\mathbf{Q}_{A_1} \quad \text{and} \quad \mathbf{Q}_{C_1' \otimes A_1}(\mathbf{C}_2' \otimes \mathbf{A}_2) = \mathbf{C}_2' \otimes \mathbf{Q}_{A_1}\mathbf{A}_2.$$

Thus

$$(\mathbf{C}_3 \otimes \mathbf{A}_3')\mathbf{Q}_{C_1' \otimes A_1}\mathbf{Q}_{Q_{C_1' \otimes A_1}(C_2' \otimes A_2)} = (\mathbf{C}_3 \otimes \mathbf{A}_3'\mathbf{Q}_{A_1})\mathbf{Q}_{C_2' \otimes Q_{A_1}A_2}.$$

From the equalities $\mathbf{P}_{(A_1:A_2)} = \mathbf{P}_{A_1} + \mathbf{P}_{Q_{A_1}A_2}$ and $\mathbf{Q}_{A_1}\mathbf{Q}_{(A_1:A_2)} = \mathbf{Q}_{(A_1:A_2)}$ we get

$$(\mathbf{C}_3 \otimes \mathbf{A}_3'\mathbf{Q}_{A_1})\mathbf{Q}_{C_2' \otimes Q_{A_1}A_2} = \mathbf{C}_3 \otimes \mathbf{A}_3'\mathbf{Q}_{(A_1:A_2)},$$

and hence (iii) is verified.

Let $i = 2$. Using the "vec" operator and by elimination of nuisance parameters (first $\mathbf{B}_1$ and then $\mathbf{B}_3$) we obtain the following estimability condition

$$\mathcal{R}\left(\sum_j \mathbf{L}_j \otimes \mathbf{K}_j'\right) \subseteq \mathcal{R}\left((\mathbf{C}_2 \otimes \mathbf{A}_2')\mathbf{Q}_{C_1' \otimes A_1}\mathbf{Q}_{Q_{C_1' \otimes A_1}(C_3' \otimes A_3)}\right).$$

Since $\mathcal{R}(\mathbf{C}_3') \subseteq \mathcal{R}(\mathbf{C}_2') \subseteq \mathcal{R}(\mathbf{C}_1')$ we have

$$(\mathbf{C}_2 \otimes \mathbf{A}_2')\mathbf{Q}_{C_1' \otimes A_1} = \mathbf{C}_2 \otimes \mathbf{A}_2'\mathbf{Q}_{A_1} \quad \text{and} \quad \mathbf{Q}_{C_1' \otimes A_1}(\mathbf{C}_3' \otimes \mathbf{A}_3) = \mathbf{C}_3' \otimes \mathbf{Q}_{A_1}\mathbf{A}_3.$$

Thus

$$(\mathbf{C}_2 \otimes \mathbf{A}_2')\mathbf{Q}_{C_1' \otimes A_1}\mathbf{Q}_{Q_{C_1' \otimes A_1}(C_3' \otimes A_3)} = (\mathbf{C}_2 \otimes \mathbf{A}_2'\mathbf{Q}_{A_1})\mathbf{Q}_{C_3' \otimes Q_{A_1}A_3} =$$
$$= \mathbf{C}_2\mathbf{Q}_{C_3'} \otimes \mathbf{A}_2'\mathbf{Q}_{A_1} + \mathbf{C}_2\mathbf{P}_{C_3'} \otimes \mathbf{A}_2'\mathbf{Q}_{(A_1:A_3)}.$$

Since $\mathcal{R}(\mathbf{X}) = (\mathbf{X}\mathbf{X}')$ we obtain

$$\mathcal{R}\left(\sum_j \mathbf{L}_j \otimes \mathbf{K}_j'\right) \subseteq \mathcal{R}\left(\mathbf{C}_2\mathbf{Q}_{C_3'}\mathbf{C}_2' \otimes \mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2 + \mathbf{C}_2\mathbf{P}_{C_3'}\mathbf{C}_2' \otimes \mathbf{A}_2'\mathbf{Q}_{(A_1:A_3)}\mathbf{A}_2\right)$$

and the nonnegative definiteness of the components in the sum implies (ii).

Let $i = 1$. Using the "vec" operator and by elimination of nuisance parameters (first $\mathbf{B}_2$ and then $\mathbf{B}_3$) we obtain the estimability condition

$$\mathcal{R}\left(\sum_j \mathbf{L}_j \otimes \mathbf{K}_j'\right) \subseteq \mathcal{R}\left((\mathbf{C}_1 \otimes \mathbf{A}_1')\mathbf{Q}_{C_2' \otimes A_2}\mathbf{Q}_{Q_{C_2' \otimes A_2}(C_3' \otimes A_3)}\right).$$

Since $\mathcal{R}(\mathbf{C}_3') \subseteq \mathcal{R}(\mathbf{C}_2') \subseteq \mathcal{R}(\mathbf{C}_1')$ we have

$$(\mathbf{C}_1 \otimes \mathbf{A}_1')\mathbf{Q}_{C_2' \otimes A_2} = \mathbf{C}_1\mathbf{Q}_{C_2'} \otimes \mathbf{A}_1' + \mathbf{C}_1\mathbf{P}_{C_2'} \otimes \mathbf{A}_1'\mathbf{Q}_{A_2} \quad \text{and} \quad \mathbf{Q}_{C_2' \otimes A_2}(\mathbf{C}_3' \otimes \mathbf{A}_3) = \mathbf{C}_3' \otimes \mathbf{Q}_{A_2}\mathbf{A}_3.$$

Thus

$$(\mathbf{C}_1 \otimes \mathbf{A}_1')\mathbf{Q}_{C_2' \otimes A_2}\mathbf{Q}_{Q_{C_2' \otimes A_2}(C_3' \otimes A_3)} = \left(\mathbf{C}_1\mathbf{Q}_{C_2'} \otimes \mathbf{A}_1' + \mathbf{C}_1\mathbf{P}_{C_2'} \otimes \mathbf{A}_1'\mathbf{Q}_{A_2}\right)\mathbf{Q}_{C_3' \otimes Q_{A_2}A_3}.$$

Using the fact that $\mathbf{P}_{C_2'}\mathbf{P}_{C_3'} = \mathbf{P}_{C_3'}$ we have $\mathbf{Q}_{C_2'}\mathbf{P}_{C_3'} = \mathbf{0}$, and from the idempotent property of $\mathbf{Q}_{A_2}$ and the property $\mathbf{P}_{Q_{A_2}A_3} = \mathbf{P}_{(A_2:A_3)} - \mathbf{P}_{A_2}$ we get

$$\left(\mathbf{C}_1\mathbf{Q}_{C_2'} \otimes \mathbf{A}_1' + \mathbf{C}_1\mathbf{P}_{C_2'} \otimes \mathbf{A}_1'\mathbf{Q}_{A_2}\right)\mathbf{Q}_{C_3' \otimes Q_{A_2}A_3}$$
$$= \mathbf{C}_1\mathbf{Q}_{C_2'} \otimes \mathbf{A}_1' + \mathbf{C}_1\mathbf{P}_{C_2'}\mathbf{Q}_{C_3'} \otimes \mathbf{A}_1'\mathbf{Q}_{A_2} + \mathbf{C}_1\mathbf{P}_{C_3'} \otimes \mathbf{A}_1'\mathbf{Q}_{(A_2:A_3)}.$$

Since $\mathcal{R}(\mathbf{X}) = \mathcal{R}(\mathbf{X}\mathbf{X}')$ we obtain

$$\mathcal{R}\left(\sum_j \mathbf{L}_j \otimes \mathbf{K}_j'\right)$$
$$\subseteq \mathcal{R}\left(\mathbf{C}_1\mathbf{Q}_{C_2'}\mathbf{C}_1' \otimes \mathbf{A}_1'\mathbf{A}_1 + \mathbf{C}_1\mathbf{P}_{C_2'}\mathbf{Q}_{C_3'}\mathbf{P}_{C_2'}\mathbf{C}_1' \otimes \mathbf{A}_1'\mathbf{Q}_{A_2}\mathbf{A}_1 + \mathbf{C}_1\mathbf{P}_{C_3'}\mathbf{C}_1' \otimes \mathbf{A}_1'\mathbf{Q}_{(A_2:A_3)}\mathbf{A}_1\right)$$

and the nonnegative definiteness of the components in the sum implies (iii). ■

Theorem 3.2. *The linear functions* $\sum_j \mathbf{K}_j\mathbf{B}_i\mathbf{L}_j$, $i=1,2,3$, *are estimable in Model II if and only if*

(i) $\mathcal{R}\left(\sum_j \mathbf{L}_j \otimes \mathbf{K}'_j\right) \subseteq \mathcal{R}\left(\mathbf{C}_1 \otimes \mathbf{A}'_1\mathbf{Q}_{A_2}\right) + \mathcal{R}\left(\mathbf{C}_1\mathbf{Q}_{C'_2} \otimes \mathbf{A}'_1\mathbf{P}_{A_2}\mathbf{Q}_{A_3}\right) + \mathcal{R}\left(\mathbf{C}_1\mathbf{Q}_{(C'_2:C'_3)} \otimes \mathbf{A}'_1\mathbf{P}_{A_3}\right)$, *for* $i=1$,

(ii) $\mathcal{R}\left(\sum_j \mathbf{L}_j \otimes \mathbf{K}'_j\right) \subseteq \mathcal{R}\left(\mathbf{C}_2\mathbf{Q}_{C'_1} \otimes \mathbf{A}'_2\mathbf{Q}_{A_3}\right) + \mathcal{R}\left(\mathbf{C}_2\mathbf{Q}_{(C'_1:C_3)} \otimes \mathbf{A}'_2\mathbf{P}_{A_3}\right)$, *for* $i=2$,

(iii) $\mathcal{R}\left(\sum_j \mathbf{L}_j \otimes \mathbf{K}'_j\right) \subseteq \mathcal{R}\left(\mathbf{C}_3\mathbf{Q}_{(C'_1:C'_2)} \otimes \mathbf{A}'_3\right)$, *for* $i=3$.

Proof. Replacing $\mathcal{R}(\mathbf{C}'_3) \subseteq \mathcal{R}(\mathbf{C}'_2) \subseteq \mathcal{R}(\mathbf{C}'_1)$ by $\mathcal{R}(\mathbf{A}_3) \subseteq \mathcal{R}(\mathbf{A}_2) \subseteq \mathcal{R}(\mathbf{A}_1)$, the proof follows similarly to the previous one. ∎

The next corollary is an immediate consequence of the theorems

Corollary 3.1. *Under Model I (Model II) with known* $\mathbf{\Sigma}$*, the least squares estimator of a linear function of* $\mathbf{B}_i$, $i=1,2,3$, *is unique if and only if the conditions of Theorem 3.1 (Theorem 3.2) are satisfied.*

Moreover, the following corollaries give some more details for uniqueness of the parameter estimators.

Corollary 3.2. *In Model I*

(i) $\widehat{\mathbf{B}}_1$ *is unique if and only if*

$$\text{rank}(\mathbf{A}_1)=m_1,\quad \text{rank}(\mathbf{C}_1)=q_1,\quad \mathcal{R}(\mathbf{A}_2)^\perp \cap \mathcal{R}(\mathbf{A}_1:\mathbf{A}_2)\cap\mathcal{R}(\mathbf{A}_2:\mathbf{A}_3)=\{\mathbf{0}\},$$
$$\mathcal{R}(\mathbf{A}_1)\cap\mathcal{R}(\mathbf{A}_2)=\{\mathbf{0}\},$$

(ii) $\widehat{\mathbf{B}}_2$ *is unique if and only if*

$$\text{rank}(\mathbf{A}_2)=m_2,\quad \text{rank}(\mathbf{C}_2)=q_2,\quad \mathcal{R}(\mathbf{A}_1)^\perp \cap \mathcal{R}(\mathbf{A}_1:\mathbf{A}_2)\cap\mathcal{R}(\mathbf{A}_2:\mathbf{A}_3)=\{\mathbf{0}\},$$
$$\mathcal{R}(\mathbf{A}_1)\cap\mathcal{R}(\mathbf{A}_2)=\{\mathbf{0}\},$$

(iii) $\widehat{\mathbf{B}}_3$ *is unique if and only if*

$$\text{rank}(\mathbf{A}_3)=m_3,\quad \text{rank}(\mathbf{C}_3)=q_3,\quad \mathcal{R}(\mathbf{A}_3)\cap\mathcal{R}(\mathbf{A}_1:\mathbf{A}_2)=\{\mathbf{0}\},$$

Corollary 3.3. *In Model II*

(i) $\widehat{\mathbf{B}}_1$ *is unique if and only if*

$$\text{rank}(\mathbf{A}_1)=m_1,\quad \text{rank}(\mathbf{C}_1)=q_1,\quad \mathcal{R}(\mathbf{C}'_2)^\perp \cap \mathcal{R}(\mathbf{C}'_1:\mathbf{C}'_2)\cap\mathcal{R}(\mathbf{C}'_2:\mathbf{C}'_3)=\{\mathbf{0}\},$$
$$\mathcal{R}(\mathbf{C}'_1)\cap\mathcal{R}(\mathbf{C}'_2)=\{\mathbf{0}\},$$

(ii) $\widehat{\mathbf{B}}_2$ *is unique if and only if*

$$\text{rank}(\mathbf{A}_2)=m_2,\quad \text{rank}(\mathbf{C}_2)=q_2,\quad \mathcal{R}(\mathbf{C}'_2)^\perp \cap \mathcal{R}(\mathbf{C}'_1:\mathbf{C}'_2)\cap\mathcal{R}(\mathbf{C}'_2:\mathbf{C}'_3)=\{\mathbf{0}\},$$
$$\mathcal{R}(\mathbf{C}'_1)\cap\mathcal{R}(\mathbf{C}'_2)=\{\mathbf{0}\},$$

(iii) $\widehat{\mathbf{B}}_3$ *is unique if and only if*

$$\text{rank}(\mathbf{A}_3)=m_3,\quad \text{rank}(\mathbf{C}_3)=q_3,\quad \mathcal{R}(\mathbf{C}'_3)\cap\mathcal{R}(\mathbf{C}'_1:\mathbf{C}'_2)=\{\mathbf{0}\},$$

4 Moments

Before considering dispersion matrices of the MLEs of the mean parameters we note that they are unbiased estimators.

Theorem 4.1. *Suppose that in Model I the MLEs* $\mathbf{K}\widehat{\mathbf{B}}_i\mathbf{L}, i = 1, 2, 3,$ *are uniquely estimated. Then* $\mathbf{K}\widehat{\mathbf{B}}_i\mathbf{L}, i = 1, 2, 3,$ *are unbiased estimators of* $\mathbf{K}\mathbf{B}_i\mathbf{L}, i = 1, 2, 3.$

Proof. $\mathbf{K}\widehat{\mathbf{B}}_3\mathbf{L}$ is unbiased since $\mathbf{S}_1$ is independent of $\mathbf{A}_3'\mathbf{Q}_{A_1:A_2}\mathbf{Y}$. $\mathbf{K}\widehat{\mathbf{B}}_2\mathbf{L}$ is unbiased since $\mathbf{S}_2$ is independent of $\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{Y}$ and $\mathcal{R}(\mathbf{C}_3') \subseteq \mathcal{R}(\mathbf{C}_2')$. $\mathbf{K}\widehat{\mathbf{B}}_1\mathbf{L}$ is unbiased since $\mathbf{S}_3$ is independent of $\mathbf{A}_1'\mathbf{Y}$ and $\mathcal{R}(\mathbf{C}_3') \subseteq \mathcal{R}(\mathbf{C}_2') \subseteq \mathcal{R}(\mathbf{C}_1')$. ■

Theorem 4.2 ([4, Theorem 4.2.6])**.** *Suppose that in Model II the MLEs* $\mathbf{K}\widehat{\mathbf{B}}_i\mathbf{L}, i = 1, 2, 3,$ *are uniquely estimated. Then* $\mathbf{K}\widehat{\mathbf{B}}_i\mathbf{L}, i = 1, 2, 3,$ *unbiased estimators of* $\mathbf{K}\mathbf{B}_i\mathbf{L}, i = 1, 2, 3.$

Theorem 4.3. *Suppose that in Model I the MLEs* $\mathbf{K}\widehat{\mathbf{B}}_i\mathbf{L}, i = 1, 2, 3,$ *are uniquely estimated. Let*

$$\gamma_1 = \frac{n-\mathrm{rank}(A_1{:}A_2{:}A_3)-1}{n-\mathrm{rank}(A_1{:}A_2{:}A_3)-p+\mathrm{rank}(C_3)-1}, \qquad \gamma_2 = \frac{p-\mathrm{rank}(C_2)}{n-\mathrm{rank}(A_1{:}A_2)-p+\mathrm{rank}(C_2)-1},$$

$$\gamma_3 = \frac{(n-\mathrm{rank}(A_1{:}A_2)-p+\mathrm{rank}(C_3)-1)(p-\mathrm{rank}(C_2))}{(n-\mathrm{rank}(A_1{:}A_2{:}A_3)-p+\mathrm{rank}(C_3)-1)(n-\mathrm{rank}(A_1{:}A_2)-p+\mathrm{rank}(C_2)-1)},$$

$$\gamma_4 = \frac{p-\mathrm{rank}(C_1)}{n-\mathrm{rank}(A_1)-p+\mathrm{rank}(C_1)-1}, \qquad \gamma_5 = \frac{n-\mathrm{rank}(A_1{:}A_2)-p+\mathrm{rank}(C_3)-1}{n-\mathrm{rank}(A_1{:}A_2{:}A_3)-p+\mathrm{rank}(C_3)-1},$$

$$\gamma_6 = \frac{n-\mathrm{rank}(A_1)-p+\mathrm{rank}(C_2)-1}{n-\mathrm{rank}(A_1{:}A_2)-p+\mathrm{rank}(C_2)-1}, \qquad \gamma_7 = \frac{n-\mathrm{rank}(A_1{:}A_2)}{n-\mathrm{rank}(A_1)},$$

$$\gamma_8 = \frac{n-\mathrm{rank}(A_1)-p+\mathrm{rank}(C_1)-1}{n-\mathrm{rank}(A_1{:}A_2)-p+\mathrm{rank}(C_1)-1}.$$

Then,

(i) *If* γ_1 *exists*

$$D[\mathbf{K}\widehat{\mathbf{B}}_3\mathbf{L}] = \gamma_1\mathbf{L}'(\mathbf{C}_3\mathbf{\Sigma}^{-1}\mathbf{C}_3')^-\mathbf{L} \otimes \mathbf{K}(\mathbf{A}_3'\mathbf{Q}_{(A_1:A_2)}\mathbf{A}_3)^-\mathbf{K}'.$$

(ii) *If* γ_1*,* γ_2 *and* γ_3 *exist*

$$\begin{aligned} D[\mathbf{K}\widehat{\mathbf{B}}_2\mathbf{L}] = {} & \gamma_1\mathbf{L}'\mathbf{C}_3'(\mathbf{C}_3\mathbf{\Sigma}^{-1}\mathbf{C}_3')^-\mathbf{C}_3\mathbf{L} \\ & \otimes\mathbf{K}(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_3(\mathbf{A}_3'\mathbf{Q}_{(A_1:A_2)}\mathbf{A}_3)^-\mathbf{A}_3'\mathbf{Q}_{A_1}\mathbf{A}_2(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{K}' \\ & + \mathbf{L}'(\mathbf{\Sigma} - (1+\gamma_2)\mathbf{\Sigma}\mathbf{C}_2'^{o}(\mathbf{C}_2'^{o'}\mathbf{\Sigma}\mathbf{C}_2'^{o})^-\mathbf{C}_2'^{o'}\mathbf{\Sigma} + \gamma_2\mathbf{\Sigma}\mathbf{C}_3'^{o}(\mathbf{C}_3'^{o'}\mathbf{\Sigma}\mathbf{C}_3'^{o})^-\mathbf{C}_3'^{o'}\mathbf{\Sigma} \\ & + \gamma_3\mathbf{C}_3'(\mathbf{C}_3\mathbf{\Sigma}^{-1}\mathbf{C}_3')^-\mathbf{C}_3)\mathbf{L} \otimes \mathbf{K}(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{K}'. \end{aligned}$$

(iii) *If* $\gamma_i,\ i = 1, \ldots, 8,$ *exist*

$$\begin{aligned} D[\mathbf{K}\widehat{\mathbf{B}}_1\mathbf{L}] = {} & \gamma_1\mathbf{L}'\mathbf{C}_3'(\mathbf{C}_3\mathbf{\Sigma}^{-1}\mathbf{C}_3')^-\mathbf{C}_3\mathbf{L} \\ & \otimes \mathbf{K}(\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{A}_1'(\mathbf{I} - \mathbf{A}_2(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1})\mathbf{A}_3(\mathbf{A}_3'\mathbf{Q}_{(A_1:A_2)}\mathbf{A}_3)^-\mathbf{A}_3' \\ & \times (\mathbf{I} - \mathbf{Q}_{A_1}\mathbf{A}_2(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2')\mathbf{A}_1(\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{K}' \\ & + \mathbf{L}'(\mathbf{\Sigma} - (1+\gamma_2)\mathbf{\Sigma}\mathbf{C}_2'^{o}(\mathbf{C}_2'^{o'}\mathbf{\Sigma}\mathbf{C}_2'^{o})^-\mathbf{C}_2'^{o'}\mathbf{\Sigma} + \gamma_2\mathbf{\Sigma}\mathbf{C}_3'^{o}(\mathbf{C}_3'^{o'}\mathbf{\Sigma}\mathbf{C}_3'^{o})^-\mathbf{C}_3'^{o'}\mathbf{\Sigma} \\ & +\gamma_3\ \mathbf{C}_3'(\mathbf{C}_3\mathbf{\Sigma}^{-1}\mathbf{C}_3')^-\mathbf{C}_3)\mathbf{L} \otimes \mathbf{K}(\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{A}_1'(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_1(\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{K}' \\ & + \mathbf{L}'\mathbf{F}_2\mathbf{L} \otimes \mathbf{K}(\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{K}', \end{aligned}$$

where

$$\begin{aligned}\mathbf{F}_2 &= \boldsymbol{\Sigma} - (1+\gamma_4)\boldsymbol{\Sigma}\mathbf{C}_1'^o(\mathbf{C}_1'^{o\prime}\boldsymbol{\Sigma}\mathbf{C}_1'^o)^-\mathbf{C}_1'^{o\prime}\boldsymbol{\Sigma} + \gamma_4\boldsymbol{\Sigma}\mathbf{C}_2'^o(\mathbf{C}_2'^{o\prime}\boldsymbol{\Sigma}\mathbf{C}_2'^o)^-\mathbf{C}_2'^{o\prime}\boldsymbol{\Sigma}\\ &\quad + \gamma_4\gamma_5\gamma_6\mathbf{C}_3'(\mathbf{C}_3\boldsymbol{\Sigma}^{-1}\mathbf{C}_3)^-\mathbf{C}_3\\ &\quad + \gamma_4\gamma_6\boldsymbol{\Sigma}\mathbf{C}_3'^o(\mathbf{C}_3'^{o\prime}\boldsymbol{\Sigma}\mathbf{C}_3'^o)^-\mathbf{C}_3'^{o\prime}\mathbf{C}_2'(\mathbf{C}_2\mathbf{C}_3'^o(\mathbf{C}_3'^{o\prime}\boldsymbol{\Sigma}\mathbf{C}_3'^o)^-\mathbf{C}_3'^{o\prime}\mathbf{C}_2')^-\mathbf{C}_2\mathbf{C}_3'^o(\mathbf{C}_3'^{o\prime}\boldsymbol{\Sigma}\mathbf{C}_3'^o)^-\mathbf{C}_3'^{o\prime}\boldsymbol{\Sigma}.\end{aligned}$$

Proof: The proofs of (i) and (ii) are given below whereas the proof of (iii), because of lengthy calculations and similarities with (ii), is presented in the Appendix.

Proof of (i): First observe that

$$\mathbf{K}(\widehat{\mathbf{B}}_3 - \mathbf{B}_3)\mathbf{L} = \mathbf{K}(\mathbf{A}_3'\mathbf{Q}_{(A_1:A_2)}\mathbf{A}_3)^-\mathbf{A}_3'\mathbf{Q}_{(A_1:A_2)}(\mathbf{Y} - E[\mathbf{Y}])\mathbf{S}_1^{-1}\mathbf{C}_3'(\mathbf{C}_3\mathbf{S}_1^{-1}\mathbf{C}_3')^-\mathbf{L}.$$

Since $\mathbf{A}_3'\mathbf{Q}_{(A_1:A_2)}\mathbf{Y}$ is independent of $\mathbf{S}_1$

$$D[\mathbf{K}\widehat{\mathbf{B}}_3\mathbf{L}] = E[\mathbf{L}'(\mathbf{C}_3\mathbf{S}_1^{-1}\mathbf{C}_3')^-\mathbf{C}_3\mathbf{S}_1^{-1}\boldsymbol{\Sigma}\mathbf{S}_1^{-1}\mathbf{C}_3'(\mathbf{C}_3\mathbf{S}_1^{-1}\mathbf{C}_3')^-\mathbf{L}] \otimes \mathbf{K}(\mathbf{A}_3'\mathbf{Q}_{(A_1:A_2)}\mathbf{A}_3)^-\mathbf{K}'.$$

However, $\mathbf{S}_1 \sim W_p(\boldsymbol{\Sigma}, n - \mathrm{rank}(\mathbf{A}_1 : \mathbf{A}_2 : \mathbf{A}_3))$ and via some calculations, see [4, (4.2.18)–(4.2.23)], the statement follows.

Proof of (ii): From Theorem 2.1 it follows that

$$\begin{aligned} D[\mathbf{K}\widehat{\mathbf{B}}_2\mathbf{L}] &= D[\mathbf{K}(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1}(\mathbf{Y} - E[\mathbf{Y}])\mathbf{S}_2^{-1}\mathbf{C}_2'(\mathbf{C}_2\mathbf{S}_2^{-1}\mathbf{C}_2')^-\mathbf{L}] \quad (4.9)\\ &\quad + D[\mathbf{K}(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_3(\widehat{\mathbf{B}}_3 - \mathbf{B}_3)\mathbf{C}_3\mathbf{S}_2^{-1}\mathbf{C}_2'(\mathbf{C}_2\mathbf{S}_2^{-1}\mathbf{C}_2')^-\mathbf{L}] \quad (4.10)\\ &\quad + Cov[\mathbf{K}(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1}(\mathbf{Y} - E[\mathbf{Y}])\mathbf{S}_2^{-1}\mathbf{C}_2'(\mathbf{C}_2\mathbf{S}_2^{-1}\mathbf{C}_2')^-\mathbf{L},\\ &\qquad \mathbf{K}(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_3(\widehat{\mathbf{B}}_3 - \mathbf{B}_3)\mathbf{C}_3\mathbf{S}_2^{-1}\mathbf{C}_2'(\mathbf{C}_2\mathbf{S}_2^{-1}\mathbf{C}_2')^-\mathbf{L}]\\ &\quad + Cov[\mathbf{K}(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_3(\widehat{\mathbf{B}}_3 - \mathbf{B}_3)\mathbf{C}_3\mathbf{S}_2^{-1}\mathbf{C}_2'(\mathbf{C}_2\mathbf{S}_2^{-1}\mathbf{C}_2')^-\mathbf{L},\\ &\qquad \mathbf{K}(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1}(\mathbf{Y} - E[\mathbf{Y}])\mathbf{S}_2^{-1}\mathbf{C}_2'(\mathbf{C}_2\mathbf{S}_2^{-1}\mathbf{C}_2')^-\mathbf{L}].\end{aligned}$$

Because of independence between $\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{Y}$ and $\mathbf{S}_1$ and between $\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{Y}$ and $\mathbf{Q}_{A_1:A_2}\mathbf{Y}$, and since $\mathcal{R}(\mathbf{C}_3') \subseteq \mathcal{R}(\mathbf{C}_2')$ implies

$$\mathbf{C}_3\mathbf{S}_2^{-1}\mathbf{C}_2'(\mathbf{C}_2\mathbf{S}_2^{-1}\mathbf{C}_2')^-\mathbf{C}_2 = \mathbf{C}_3$$

we obtain

$$\begin{aligned} &Cov[\mathbf{K}(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1}(\mathbf{Y} - E[\mathbf{Y}])\mathbf{S}_2^{-1}\mathbf{C}_2'(\mathbf{C}_2\mathbf{S}_2^{-1}\mathbf{C}_2')^-\mathbf{L},\\ &\qquad \mathbf{K}(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_3(\widehat{\mathbf{B}}_3 - \mathbf{B}_3)\mathbf{C}_3\mathbf{S}_2^{-1}\mathbf{C}_2'(\mathbf{C}_2\mathbf{S}_2^{-1}\mathbf{C}_2')^-\mathbf{L}]\\ = \; &Cov[\mathbf{K}(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1}(\mathbf{Y} - E[\mathbf{Y}])\mathbf{C}_2'(\mathbf{C}_2\mathbf{C}_2')^-\mathbf{L},\\ &\qquad \mathbf{K}(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_3(\widehat{\mathbf{B}}_3 - \mathbf{B}_3)\mathbf{C}_3\mathbf{C}_2'(\mathbf{C}_2\mathbf{C}_2')^-\mathbf{L}] = \mathbf{0},\end{aligned}$$

where the last equality follows because $\widehat{\mathbf{B}}_3$ is unbiased. Thus, $D[\mathbf{K}\widehat{\mathbf{B}}_2\mathbf{L}]$ equals the sum of (4.9) and (4.10) and we are going to consider these terms separately. However, we immediately obtain from (i) that (4.10) equals

$$\begin{aligned} &D[\mathbf{K}(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_3(\widehat{\mathbf{B}}_3 - \mathbf{B}_3)\mathbf{C}_3\mathbf{S}_2^{-1}\mathbf{C}_2'(\mathbf{C}_2\mathbf{S}_2^{-1}\mathbf{C}_2')^-\mathbf{L}]\\ &\quad = \gamma_1\mathbf{L}'\mathbf{C}_3'(\mathbf{C}_3\boldsymbol{\Sigma}^{-1}\mathbf{C}_3')^-\mathbf{C}_3\mathbf{L}\\ &\qquad \otimes\mathbf{K}(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_3(\mathbf{A}_3'\mathbf{Q}_{(A_1:A_2)}\mathbf{A}_3)^-\mathbf{A}_3'\mathbf{Q}_{A_1}\mathbf{A}_2(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{K}'. \quad (4.11)\end{aligned}$$

Now (4.9) is exploited. The independence between $\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{Y}$ and $\mathbf{S}_2$ yields that (4.9) equals

$$E[\mathbf{L}'(\mathbf{C}_2\mathbf{S}_2^{-1}\mathbf{C}_2')^-\mathbf{C}_2\mathbf{S}_2^{-1}\boldsymbol{\Sigma}\mathbf{S}_2^{-1}\mathbf{C}_2'(\mathbf{C}_2\mathbf{S}_2^{-1}\mathbf{C}_2')^-\mathbf{L}] \otimes \mathbf{K}(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{K}'. \tag{4.12}$$

The expectation in (4.12) will be considered in detail and

$$\mathbf{F}_1 = E[\mathbf{P}_{C_2';S_2^{-1}}\boldsymbol{\Sigma}\mathbf{P}'_{C_2';S_2^{-1}}], \tag{4.13}$$

is introduced since via $\mathbf{L}'(\mathbf{C}_2\mathbf{C}_2')^-\mathbf{C}_2\mathbf{F}_1\mathbf{C}_2'(\mathbf{C}_2\mathbf{C}_2')^-\mathbf{L}$ the expectation in (4.12) is obtained. Observe, that $\mathbf{S}_2$ can be rewritten as

$$\mathbf{S}_2 = \mathbf{S}_1 + \mathbf{S}_1\mathbf{C'}_3^o(\mathbf{C'}_3^{o'}\mathbf{S}_1\mathbf{C'}_3^o)^-\mathbf{C'}_3^{o'}\mathbf{Y}\mathbf{P}_{Q_{(A_1:A_2)}A_3}\mathbf{Y}'\mathbf{C'}_3^o(\mathbf{C'}_3^{o'}\mathbf{S}_1\mathbf{C'}_3^o)^-\mathbf{C'}_3^{o'}\mathbf{S}_1$$

and, following [4, pp. 376], formula (4.13) can be expressed as

$$\begin{aligned}
\mathbf{F}_1 &= E[(\mathbf{I}_p - \mathbf{S}_2\mathbf{C'}_2^o(\mathbf{C'}_2^{o'}\mathbf{S}_2\mathbf{C'}_2^o)^-\mathbf{C'}_2^{o'})\boldsymbol{\Sigma}(\mathbf{I}_p - \mathbf{C'}_2^o(\mathbf{C'}_2^{o'}\mathbf{S}_2\mathbf{C'}_2^o)^-\mathbf{C'}_2^{o'}\mathbf{S}_2)] \\
&= \boldsymbol{\Sigma} - E[\mathbf{S}_2\mathbf{C'}_2^o(\mathbf{C'}_2^{o'}\mathbf{S}_2\mathbf{C'}_2^o)^-\mathbf{C'}_2^{o'}\boldsymbol{\Sigma}] - E[\boldsymbol{\Sigma}\mathbf{C'}_2^o(\mathbf{C'}_2^{o'}\mathbf{S}_2\mathbf{C'}_2^o)^-\mathbf{C'}_2^{o'}\mathbf{S}_2] \\
&\quad + E[\mathbf{S}_2\mathbf{C'}_2^o(\mathbf{C'}_2^{o'}\mathbf{S}_2\mathbf{C'}_2^o)^-\mathbf{C'}_2^{o'}\boldsymbol{\Sigma}\mathbf{C'}_2^o(\mathbf{C'}_2^{o'}\mathbf{S}_2\mathbf{C'}_2^o)^-\mathbf{C'}_2^{o'}\mathbf{S}_2] \\
&= \boldsymbol{\Sigma} - \boldsymbol{\Sigma}^{1/2}E[\boldsymbol{\Sigma}^{-1/2}\mathbf{S}_2\mathbf{C'}_2^o(\mathbf{C'}_2^{o'}\mathbf{S}_2\mathbf{C'}_2^o)^-\mathbf{C'}_2^{o'}\boldsymbol{\Sigma}^{1/2}]\boldsymbol{\Sigma}^{1/2} \\
&\quad - \boldsymbol{\Sigma}^{1/2}E[\boldsymbol{\Sigma}^{1/2}\mathbf{C'}_2^o(\mathbf{C'}_2^{o'}\mathbf{S}_2\mathbf{C'}_2^o)^-\mathbf{C'}_2^{o'}\mathbf{S}_2\boldsymbol{\Sigma}^{-1/2}]\boldsymbol{\Sigma}^{1/2} \\
&\quad + \boldsymbol{\Sigma}^{1/2}E[\boldsymbol{\Sigma}^{-1/2}\mathbf{S}_2\mathbf{C'}_2^o(\mathbf{C'}_2^{o'}\mathbf{S}_2\mathbf{C'}_2^o)^-\mathbf{C'}_2^{o'}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{1/2}\mathbf{C'}_2^o(\mathbf{C'}_2^{o'}\mathbf{S}_2\mathbf{C'}_2^o)^-\mathbf{C'}_2^{o'}\mathbf{S}_2\boldsymbol{\Sigma}^{-1/2}]\boldsymbol{\Sigma}^{1/2}.
\end{aligned} \tag{4.14}$$

Put

$$\begin{aligned}
&\mathbf{V}_1 = \boldsymbol{\Sigma}^{-1/2}\mathbf{S}_1\boldsymbol{\Sigma}^{-1/2}, \qquad \mathbf{D}_2^o = \boldsymbol{\Sigma}^{1/2}\mathbf{C'}_2^o, \qquad \mathbf{Z} = \mathbf{Y}\boldsymbol{\Sigma}^{-1/2}, \qquad \mathbf{D}_3^o = \boldsymbol{\Sigma}^{1/2}\mathbf{C'}_3^o, \\
&\mathbf{V}_2 = \mathbf{V}_1 + \mathbf{V}_1\mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{V}_1\mathbf{D}_3^o)^-\mathbf{D}_3^{o'}\mathbf{Z}'\mathbf{P}_{Q_{(A_1:A_2)}A_3}\mathbf{Z}\mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{V}_1\mathbf{D}_3^o)^-\mathbf{D}_3^{o'}\mathbf{V}_1.
\end{aligned}$$

Then, since $\mathcal{R}(\mathbf{D}_2)^\perp \subseteq \mathcal{R}(\mathbf{D}_3)^\perp$ equation (4.14) can be written

$$\begin{aligned}
\mathbf{F}_1 &= \boldsymbol{\Sigma} - \boldsymbol{\Sigma}^{1/2}E[\mathbf{V}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}]\boldsymbol{\Sigma}^{1/2} - \boldsymbol{\Sigma}^{1/2}E[\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}\mathbf{V}_2]\boldsymbol{\Sigma}^{1/2} \\
&\quad + \boldsymbol{\Sigma}^{1/2}E[\mathbf{V}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}\mathbf{V}_2]\boldsymbol{\Sigma}^{1/2} \\
&= \boldsymbol{\Sigma} - \boldsymbol{\Sigma}^{1/2}E[\mathbf{T}_1]\boldsymbol{\Sigma}^{1/2} - \boldsymbol{\Sigma}^{1/2}E[\mathbf{T}_1']\boldsymbol{\Sigma}^{1/2} + \boldsymbol{\Sigma}^{1/2}E[\mathbf{T}_1\mathbf{T}_1']\boldsymbol{\Sigma}^{1/2},
\end{aligned} \tag{4.15}$$

where

$$\mathbf{T}_1 = \mathbf{V}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}. \tag{4.16}$$

Firstly $E[\mathbf{T}_1]$ is obtained. Since $\mathbf{I} = \mathbf{P}_{D_3^o} + \mathbf{P}_{D_3}$

$$E[\mathbf{T}_1] = \mathbf{P}_{D_3^o}E[\mathbf{T}_1] + \mathbf{P}_{D_3}E[\mathbf{T}_1]. \tag{4.17}$$

Using $\mathcal{R}(\mathbf{D}_2)^\perp \subseteq \mathcal{R}(\mathbf{D}_3)^\perp$ it can be observed that

$$\begin{aligned}
\mathbf{D}_3^{o'}\mathbf{V}_2\mathbf{D}_2^o &= \mathbf{D}_3^{o'}\mathbf{V}_1\mathbf{D}_2^o + \mathbf{D}_3^{o'}\mathbf{V}_1\mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{V}_1\mathbf{D}_3^o)^-\mathbf{D}_3^{o'}\mathbf{Z}'\mathbf{P}_{Q_{(A_1:A_2)}A_3}\mathbf{Z}\mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{V}_1\mathbf{D}_3^o)^-\mathbf{D}_3^{o'}\mathbf{V}_1\mathbf{D}_2^o \\
&= \mathbf{D}_3^{o'}(\mathbf{V}_1 + \mathbf{Z}'\mathbf{P}_{Q_{(A_1:A_2)}A_3}\mathbf{Z})\mathbf{D}_2^o \equiv \mathbf{D}_3^{o'}\mathbf{W}_2\mathbf{D}_2^o
\end{aligned} \tag{4.18}$$

and similarly

$$\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{D}_2^o = \mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o, \tag{4.19}$$

where because of independence between $\mathbf{V}_1$ and $\mathbf{Z}'\mathbf{P}_{Q_{(A_1:A_2)}A_3}\mathbf{Z}$

$$\mathbf{W}_2 \sim \mathcal{W}_p(\mathbf{I}_p, n - \mathrm{rank}(\mathbf{A}_1 : \mathbf{A}_2)).$$

Let us in (4.17) determine $\mathbf{P}_{D_3^o}E[\mathbf{T}_1]$:

$$\begin{aligned}\mathbf{P}_{D_3^o}E[\mathbf{T}_1] &= \mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{D}_3^o)^- E[\mathbf{D}_3^{o'}\mathbf{V}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}] \\ &= \mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{D}_3^o)^- E[\mathbf{D}_3^{o'}\mathbf{W}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}] = \mathbf{P}_{D_3^o}E[\mathbf{W}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}] \\ &= \mathbf{P}_{D_3^o}\mathbf{P}_{D_2^o} = \mathbf{P}_{D_2^o} = \mathbf{P}_{\Sigma^{1/2}C_2'^o}, \end{aligned} \tag{4.20}$$

(for the last taken expectation see [4, pp. 275]).

Moreover, in (4.17) consider $\mathbf{P}_{D_3}E[\mathbf{T}_1]$. Since $\mathbf{V}_1$ is Wishart distributed we can factorize it as $\mathbf{V}_1 = \mathbf{X}\mathbf{X}'$, where $\mathbf{X} \sim N_{p,n}(\mathbf{0}, \mathbf{I}_p, \mathbf{I}_n)$. Furthermore, since $\mathbf{D}_3'(\mathbf{D}_2^o : \mathbf{D}_3^o) = \mathbf{0}$ we have that $\mathbf{D}_3'\mathbf{X}$ is independent of $\mathbf{D}_2^{o'}\mathbf{X}$ and $\mathbf{D}_3^{o'}\mathbf{X}$, and hence

$$\begin{aligned}&\mathbf{P}_{D_3}E[\mathbf{T}_1] = \mathbf{D}_3(\mathbf{D}_3'\mathbf{D}_3)^- E[\mathbf{D}_3'\mathbf{V}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}] \\ &\quad = \mathbf{D}_3(\mathbf{D}_3'\mathbf{D}_3)^- E[\mathbf{D}_3'\mathbf{V}_1\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'} \\ &\qquad +\mathbf{D}_3'\mathbf{V}_1\mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{W}_2\mathbf{D}_3^o)^-\mathbf{D}_3^{o'}\mathbf{Z}'\mathbf{P}_{Q_{(A_1:A_2)}A_3}\mathbf{Z}\mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{W}_2\mathbf{D}_3^o)^-\mathbf{D}_3^{o'}\mathbf{V}_1\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}] \\ &\quad = \mathbf{D}_3(\mathbf{D}_3'\mathbf{D}_3)^- E[\mathbf{D}_3'\mathbf{X}(\mathbf{I}_n + \mathbf{X}'\mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{W}_2\mathbf{D}_3^o)^-\mathbf{D}_3^{o'}\mathbf{Z}'\mathbf{P}_{Q_{(A_1:A_2)}A_3}\mathbf{Z}\mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{W}_2\mathbf{D}_3^o)^-\mathbf{D}_3^{o'}\mathbf{X})\mathbf{X}'\mathbf{D}_2^o \\ &\qquad \times(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}] \\ &\quad = \mathbf{D}_3(\mathbf{D}_3'\mathbf{D}_3)^- \underbrace{E[\mathbf{D}_3'\mathbf{X}]}_{\mathbf{0}} \\ &\qquad \times E[(\mathbf{I}_n + \mathbf{X}'\mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{W}_2\mathbf{D}_3^o)^-\mathbf{D}_3^{o'}\mathbf{Z}'\mathbf{P}_{Q_{(A_1:A_2)}A_3}\mathbf{Z}\mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{W}_2\mathbf{D}_3^o)^-\mathbf{D}_3^{o'}\mathbf{X})\mathbf{X}'\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}] \\ &\qquad = \mathbf{0}.\end{aligned}$$

Thus,

$$E[\mathbf{T}_1] = \mathbf{P}_{\Sigma^{1/2}C_2'^o}\ . \tag{4.21}$$

In the next we consider $E[\mathbf{T}_1\mathbf{T}_1']$. Since again $\mathbf{I} = \mathbf{P}_{D_3^o} + \mathbf{P}_{D_3}$ and since $\mathbf{D}_3'\mathbf{X}$ is independent of $\mathbf{D}_2^{o'}\mathbf{X}$ and $\mathbf{D}_3^{o'}\mathbf{X}$, we will calculate

$$(\mathbf{P}_{D_3^o} + \mathbf{P}_{D_3})E[\mathbf{T}_1\mathbf{T}_1'](\mathbf{P}_{D_3^o} + \mathbf{P}_{D_3}) = \mathbf{P}_{D_3^o}E[\mathbf{T}_1\mathbf{T}_1']\mathbf{P}_{D_3^o} + \mathbf{P}_{D_3}E[\mathbf{T}_1\mathbf{T}_1']\mathbf{P}_{D_3},$$

since $\mathbf{P}_{D_3^o}E[\mathbf{T}_1\mathbf{T}_1']\mathbf{P}_{D_3} = \mathbf{0}$. Using (4.18) and (4.19) we can write

$$\begin{aligned}&\mathbf{P}_{D_3^o}E[\mathbf{T}_1\mathbf{T}_1']\mathbf{P}_{D_3^o} \\ &\quad = \mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{D}_3^o)^- E\left[\mathbf{D}_3^{o'}\mathbf{V}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{D}_3^o\right](\mathbf{D}_3^{o'}\mathbf{D}_3^o)^-\mathbf{D}_3^{o'} \\ &\quad = \mathbf{P}_{D_3^o}E[\mathbf{W}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}\mathbf{W}_2]\mathbf{P}_{D_3^o}\end{aligned}$$

and therefore, following [4, pp. 419, formula (4.2.51)], we obtain

$$\mathbf{P}_{D_3^o}E[\mathbf{T}_1\mathbf{T}_1']\mathbf{P}_{D_3^o} = (1-\gamma_2)\mathbf{P}_{\Sigma^{1/2}C_2'^o} + \gamma_2\mathbf{P}_{\Sigma^{1/2}C_3'^o}, \tag{4.22}$$

since $\mathbf{D}_2 = \mathbf{\Sigma}^{-1/2}\mathbf{C}_2'$, $\mathbf{D}_3 = \mathbf{\Sigma}^{-1/2}\mathbf{C}_3'$ and where γ_2 is defined in the statement of the theorem.

In order to verify (ii) it remains to calculate $\mathbf{P}_{D_3}E[\mathbf{T}_1\mathbf{T}_1']\mathbf{P}_{D_3}$. Using $\mathbf{V}_1 = \mathbf{X}\mathbf{X}'$ and the independence of $\mathbf{D}_3'\mathbf{X}$ with $\mathbf{D}_2^{o'}\mathbf{X}$ and $\mathbf{D}_3^{o'}\mathbf{X}$, we obtain

$$\begin{aligned}
&\mathbf{P}_{D_3}E[\mathbf{T}_1\mathbf{T}_1']\mathbf{P}_{D_3}\\
&\quad= \mathbf{D}_3(\mathbf{D}_3'\mathbf{D}_3)^{-}E[\mathbf{D}_3'\mathbf{V}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{D}_2^o)^{-}\mathbf{D}_2^{o'}\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{D}_2^o)^{-}\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{D}_3](\mathbf{D}_3'\mathbf{D}_3)^{-}\mathbf{D}_3'\\
&\quad= E[\mathrm{tr}\{\mathbf{V}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^{-}\mathbf{D}_2^{o'}\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^{-}\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{V}_1^{-1}\}]\mathbf{P}_{D_3}.
\end{aligned}$$

Moreover,

$$\mathbf{V}_1^{-1} = \mathbf{V}_1^{-1}\mathbf{D}_3(\mathbf{D}_3'\mathbf{V}_1^{-1}\mathbf{D}_3)^{-}\mathbf{D}_3'\mathbf{V}_1^{-1} + \mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{V}_1\mathbf{D}_3^o)^{-}\mathbf{D}_3^{o'}.$$

Since $\mathbf{D}_3'\mathbf{V}_1^{-1}\mathbf{V}_2\mathbf{D}_2^o = \mathbf{0}$, we get

$$\begin{aligned}
&\mathbf{P}_{D_3}E[\mathbf{T}_1\mathbf{T}_1']\mathbf{P}_{D_3}\\
&\quad= E[\mathrm{tr}\{\mathbf{D}_3^{o'}\mathbf{W}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^{-}\mathbf{D}_2^{o'}\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^{-}\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{V}_1\mathbf{D}_3^o)^{-}\}]\mathbf{P}_{D_3}. \quad (4.23)
\end{aligned}$$

Without loss of generality from now on we identify $\mathbf{D}_2^o$ and $\mathbf{D}_3^o$ with matrices of full rank, i.e. $\mathbf{D}_2^o$: $p\times(p-\mathrm{rank}(\mathbf{D}_2))$ and $\mathbf{D}_3^o$: $p\times(p-\mathrm{rank}(\mathbf{D}_3))$. We are going to rewrite (4.23) in a canonical form and use that there exist a non-singular matrix $\mathbf{M}:\ (p-\mathrm{rank}(\mathbf{D}_3))\times(p-\mathrm{rank}(\mathbf{D}_3))$ and an orthogonal matrix $\mathbf{\Gamma}$: $p\times p$ such that

$$\mathbf{D}_3^{o'} = \mathbf{M}(\mathbf{I}_r : \mathbf{0})\mathbf{\Gamma}, \qquad r = p - \mathrm{rank}(\mathbf{D}_3) = p - \mathrm{rank}(\mathbf{C}_3)$$

and since $\mathcal{R}(\mathbf{D}_2)^{\perp} \subseteq \mathcal{R}(\mathbf{D}_3)^{\perp}$ there exists a matrix $\mathbf{Q}$ such that

$$\mathbf{D}_2^o = \mathbf{\Gamma}'(\mathbf{I}_r : \mathbf{0})'\mathbf{M}'\mathbf{Q}.$$

Therefore the trace in (4.23) equals

$$\mathrm{tr}\{\mathbf{W}_{11}\mathbf{M}'\mathbf{Q}(\mathbf{Q}'\mathbf{M}\mathbf{W}_{11}\mathbf{M}'\mathbf{Q})^{-1}\mathbf{Q}'\mathbf{M}\mathbf{M}'\mathbf{Q}(\mathbf{Q}'\mathbf{M}\mathbf{W}_{11}\mathbf{M}'\mathbf{Q})^{-1}\mathbf{Q}'\mathbf{M}\mathbf{W}_{11}(\mathbf{V}_{11})^{-1}\}, \qquad (4.24)$$

where $\mathbf{W}_{11} = (\mathbf{I}_r : \mathbf{0})\mathbf{\Gamma}\mathbf{W}_2\mathbf{\Gamma}'(\mathbf{I}_r : \mathbf{0})'$ and $\mathbf{V}_{11} = (\mathbf{I}_r : \mathbf{0})\mathbf{\Gamma}\mathbf{V}_1\mathbf{\Gamma}'(\mathbf{I}_r : \mathbf{0})'$. The next lemma will be applied several times in the subsequent and the proof can be found in [4, Theorem 2.4.8, pp. 248-250, Theorem 2.4.15, pp. 263].

Lemma 4.1. *Let $\mathbf{V} \sim W_p(\mathbf{I}, n)$, $p < n$, and $\mathbf{W} \sim W_p(\mathbf{I}, m)$, $p < m$. Then,*

$$\mathbf{B} = (\mathbf{V}+\mathbf{W})^{-1/2}\mathbf{V}(\mathbf{V}+\mathbf{W})^{-1/2}$$

is multivariate beta type I distributed and $\mathbf{B}$ is independent of $\mathbf{V}+\mathbf{W}$. Moreover,

$$\begin{aligned}
E[\mathbf{B}] &= \tfrac{n}{m+n}\mathbf{I}_p,\\
E[\mathbf{B}^{-1}] &= \tfrac{m+n-p-1}{n-p-1}\mathbf{I}_p, \qquad n-p-1 > 0.
\end{aligned}$$

Put in (4.24)

$$\mathbf{N} = \mathbf{W}_{11}^{1/2}(\mathbf{V}_{11})^{-1}\mathbf{W}_{11}^{1/2}$$

which by Lemma 4.1 follows an inverse multivariate beta type I distribution with the important fact that the distribution is independent of $\mathbf{W}_{11}$ and

$$E[\mathbf{N}] = \tfrac{n-\mathrm{rank}(A_1:A_2)-r-1}{n-\mathrm{rank}(A_1:A_2:A_3)-r-1}\mathbf{I}_r,$$

since $\mathbf{W}_{11} \sim W_r(\mathbf{I}_r, n - \text{rank}(\mathbf{A}_1 : \mathbf{A}_2))$ and $\mathbf{V}_{11} \sim W_r(\mathbf{I}_r, n - \text{rank}(\mathbf{A}_1 : \mathbf{A}_2 : \mathbf{A}_3))$.

Hence,

$$\mathbf{P}_{D_3} E[\mathbf{T}_1 \mathbf{T}_1'] \mathbf{P}_{D_3} = \gamma_3 \mathbf{P}_{\Sigma^{-1/2} C_3'},$$

since $\mathbf{D}_3 = \mathbf{\Sigma}^{-1/2} \mathbf{C}_3'$ and γ_3 is defined in the formulation of the theorem. Thus,

$$E[\mathbf{T}_1 \mathbf{T}_1'] = (1 - \gamma_2) \mathbf{P}_{\Sigma^{1/2} C_2'^o} + \gamma_2 \mathbf{P}_{\Sigma^{1/2} C_3'^o} + \gamma_3 \mathbf{P}_{\Sigma^{-1/2} C_3'} \tag{4.25}$$

and

$$\begin{aligned} \mathbf{F}_1 = \mathbf{\Sigma} - (1 + \gamma_2) \mathbf{\Sigma} \mathbf{C}_2'^o (\mathbf{C}_2'^o \mathbf{\Sigma} \mathbf{C}_2'^o)^- \mathbf{C}_2'^o \mathbf{\Sigma} \quad &+ \quad \gamma_2 \mathbf{\Sigma} \mathbf{C}_3'^o (\mathbf{C}_3^{o'} \mathbf{\Sigma} \mathbf{C}_3'^o)^- \mathbf{C}_3^{o'} \mathbf{\Sigma} \\ &+ \quad \gamma_3 \mathbf{C}_3' (\mathbf{C}_3 \mathbf{\Sigma}^{-1} \mathbf{C}_3')^- \mathbf{C}_3. \end{aligned} \tag{4.26}$$

Finally, from (4.11) and (4.26) statement (ii) of the theorem is obtained. ■

The next theorem can be found in [4, Theorem 4.2.11 (iii)].

Theorem 4.4. *For Model II let $\widehat{\mathbf{B}}_i$, $i = 1, 2, 3$, be given in Theorem 2.2 and suppose that for each $\widehat{\mathbf{B}}_i$ the uniqueness conditions in Theorem 3.2 are satisfied. Then, if the dispersion matrices are supposed to exist,*

$$D[\widehat{\mathbf{B}}_3] = \tfrac{n - m_3 - 1}{n - m_3 - v_2 + q_3 - 1} (\mathbf{A}_3' \mathbf{A}_3)^{-1} \otimes (\mathbf{C}_3 \mathbf{G}_2 (\mathbf{G}_2' \mathbf{\Sigma} \mathbf{G}_2)^{-1} \mathbf{G}_2' \mathbf{C}_3')^{-1},$$

$$\begin{aligned} D[\widehat{\mathbf{B}}_2] &= (\mathbf{A}_2' \mathbf{A}_2)^{-1} \mathbf{A}_2' (\mathbf{P}_{A_2} - \mathbf{P}_{A_3}) \mathbf{A}_2 (\mathbf{A}_2' \mathbf{A}_2)^{-1} \otimes \tfrac{n - m_2 - 1}{n - \text{rank}(A_1) - v_1 + q_2 - 1} (\mathbf{C}_2 \mathbf{G}_1 (\mathbf{G}_1' \mathbf{\Sigma} \mathbf{G}_1)^{-1} \mathbf{G}_1' \mathbf{C}_2')^{-1} \\ &\quad + (\mathbf{A}_2' \mathbf{A}_2)^{-1} \mathbf{A}_2' \mathbf{P}_{A_3} \mathbf{A}_2 (\mathbf{A}_2' \mathbf{A}_2)^{-1} \otimes \Big\{ \tfrac{n - \text{rank}(A_3) - 1}{n - \text{rank}(A_3) - v_3 - 1} \mathbf{W}_1 \mathbf{C}_3' (\mathbf{C}_3 \mathbf{G}_2 (\mathbf{G}_2' \mathbf{\Sigma} \mathbf{G}_2)^{-1} \mathbf{G}_2' \mathbf{C}_3')^{-1} \mathbf{C}_3 \mathbf{W}_1' \\ &\qquad + (1 + \tfrac{k_2 v_3}{n - \text{rank}(A_3) - v_3 - 1}) (\mathbf{C}_2 \mathbf{G}_1 (\mathbf{G}_1' \mathbf{\Sigma} \mathbf{G}_1)^{-1} \mathbf{G}_1' \mathbf{C}_2')^{-1} \Big\}, \end{aligned}$$

$$\begin{aligned} D[\widehat{\mathbf{B}}_1] &= (\mathbf{A}_1' \mathbf{A}_1)^{-1} \mathbf{A}_1' (\mathbf{P}_{A_1} - \mathbf{P}_{A_2}) \mathbf{A}_1 (\mathbf{A}_1' \mathbf{A}_1)^{-1} \otimes \tfrac{n - m_1 - 1}{n - m_1 - p + q_1 - 1} (\mathbf{C}_1 \mathbf{\Sigma}^{-1} \mathbf{C}_1')^{-1} \\ &\quad + (\mathbf{A}_1' \mathbf{A}_1)^{-1} \mathbf{A}_1' (\mathbf{P}_{A_2} - \mathbf{P}_{A_3}) \mathbf{A}_1 (\mathbf{A}_1' \mathbf{A}_1)^{-1} \otimes \Big\{ (1 + \tfrac{k_1 v_2}{n - \text{rank}(A_2) - v_2 - 1}) (\mathbf{C}_1 \mathbf{\Sigma}^{-1} \mathbf{C}_1')^{-1} \\ &\qquad + \tfrac{n - \text{rank}(A_2) - 1}{n - \text{rank}(A_2) - v_2 - 1} \mathbf{W}_2 \mathbf{C}_2' (\mathbf{C}_2 \mathbf{G}_1 (\mathbf{G}_1' \mathbf{\Sigma} \mathbf{G}_1)^{-1} \mathbf{G}_1' \mathbf{C}_2')^{-1} \mathbf{C}_2 \mathbf{W}_2' \Big\} \\ &\quad + (\mathbf{A}_1' \mathbf{A}_1)^{-1} \mathbf{A}_1' \mathbf{P}_{A_3} \mathbf{A}_1 (\mathbf{A}_1' \mathbf{A}_1)^{-1} \otimes \Big\{ (1 + \tfrac{k_1 k_2 v_3}{n - \text{rank}(A_3) - v_3 - 1}) (\mathbf{C}_1 \mathbf{\Sigma}^{-1} \mathbf{C}_1')^{-1} \\ &\qquad + \mathbf{W}_2 \mathbf{C}_2' (\mathbf{C}_2 \mathbf{G}_1 (\mathbf{G}_1' \mathbf{\Sigma} \mathbf{G}_1)^{-1} \mathbf{G}_1' \mathbf{C}_2')^{-1} \mathbf{C}_2 \mathbf{W}_2' \\ &\qquad + \tfrac{n - \text{rank}(A_3) - 1}{n - \text{rank}(A_3) - v_3 - 1} \mathbf{W}_3 \mathbf{C}_3' (\mathbf{C}_3 \mathbf{G}_2 (\mathbf{G}_2' \mathbf{\Sigma} \mathbf{G}_2)^{-1} \mathbf{G}_2' \mathbf{C}_3')^{-1} \mathbf{C}_3 \mathbf{W}_3' \Big\}, \end{aligned}$$

with

$$v_1 = p - \text{rank}(\mathbf{C}_1), \quad v_i = p - \text{rank}(\mathbf{C}_1' : \cdots : \mathbf{C}_i') + \text{rank}(\mathbf{C}_1' : \cdots : \mathbf{C}_{i-1}'), \quad i = 2, 3,$$

$$k_j = \tfrac{n - \text{rank}(A_{j+1}) - v_j - 1}{n - \text{rank}(A_j) - v_j - 1}, \quad j = 1, 2,$$

$$\mathbf{G}_1 = (\mathbf{C}_1')^o, \qquad \mathbf{G}_2 = \mathbf{G}_1 (\mathbf{G}_1^{o'} \mathbf{C}_2)^o,$$

$$\mathbf{W}_1 = (\mathbf{C}_2 \mathbf{G}_1 (\mathbf{G}_1' \mathbf{\Sigma} \mathbf{G}_1)^{-1} \mathbf{G}_1' \mathbf{C}_2')^{-1} \mathbf{C}_2 \mathbf{G}_1 (\mathbf{G}_1' \mathbf{\Sigma} \mathbf{G}_1)^{-1} \mathbf{G}_1',$$

$$\mathbf{W}_2 = (\mathbf{C}_1 \mathbf{\Sigma}^{-1} \mathbf{C}_1')^{-1} \mathbf{C}_1 \mathbf{\Sigma}^{-1}, \qquad \mathbf{W}_3 = \mathbf{W}_2 (\mathbf{I}_p - \mathbf{C}_2' \mathbf{W}_1).$$

Next theorems give $E(n\widehat{\mathbf{\Sigma}})$ for Model I and II, respectively.

Theorem 4.5. *For Model I let $\widehat{\boldsymbol{\Sigma}}$ be given in Theorem 2.1. Then,*

$$\begin{aligned} E[n\widehat{\boldsymbol{\Sigma}}] &= (n-\mathrm{rank}(\mathbf{A}_1:\mathbf{A}_2:\mathbf{A}_3))\boldsymbol{\Sigma} \\ &\quad +(\mathrm{rank}(\mathbf{A}_1:\mathbf{A}_2:\mathbf{A}_3)-\mathrm{rank}(\mathbf{A}_1:\mathbf{A}_2))\left(\boldsymbol{\Sigma}-(1-\gamma_9)\mathbf{C}_3'(\mathbf{C}_3\boldsymbol{\Sigma}\mathbf{C}_3')^-\mathbf{C}_3\right) \\ &\quad +(\mathrm{rank}(\mathbf{A}_1:\mathbf{A}_2)-\mathrm{rank}(\mathbf{A}_1))\boldsymbol{\Sigma}^{1/2}E[\mathbf{T}_1\mathbf{T}_1']\boldsymbol{\Sigma}^{1/2}+\mathrm{rank}(\mathbf{A}_1)\boldsymbol{\Sigma}^{1/2}E[\mathbf{T}_2\mathbf{T}_2']\boldsymbol{\Sigma}^{1/2}, \end{aligned}$$

where $\gamma_9=\frac{p-\mathrm{rank}(C_3)}{n-\mathrm{rank}(A_1:A_2:A_3)-p+\mathrm{rank}(C_3))-1}$ *and* $E[\mathbf{T}_1\mathbf{T}_1']$ *and* $E[\mathbf{T}_2\mathbf{T}_2']$ *are given by (4.25) and (A-21), respectively.*

Proof. The expression follows from the following calculations:

$$\begin{aligned} E[n\widehat{\boldsymbol{\Sigma}}] &= E[\mathbf{S}_1]+\mathrm{rank}(\mathbf{Q}_{A_1:A_2}\mathbf{A}_3)E[\mathbf{Q}_{C_3';S_1^{-1}}\boldsymbol{\Sigma}\mathbf{Q}_{C_3';S_1^{-1}}']+\mathrm{rank}(\mathbf{Q}_{A_1}\mathbf{A}_2)E[\mathbf{Q}_{C_2';S_2^{-1}}\boldsymbol{\Sigma}\mathbf{Q}_{C_2';S_2^{-1}}'] \\ &\quad +\mathrm{rank}(\mathbf{A}_1)E[\mathbf{Q}_{C_1;S_3^{-1}}\boldsymbol{\Sigma}\mathbf{Q}_{C_1';S_3^{-1}}'] \\ &= E[\mathbf{S}_1]+\mathrm{rank}(\mathbf{Q}_{A_1:A_2}\mathbf{A}_3)E[\mathbf{Q}_{C_3';S_1^{-1}}\boldsymbol{\Sigma}\mathbf{Q}_{C_3';S_1^{-1}}']+\mathrm{rank}(\mathbf{Q}_{A_1}\mathbf{A}_2)\boldsymbol{\Sigma}^{1/2}E[\mathbf{T}_1\mathbf{T}_1']\boldsymbol{\Sigma}^{1/2} \\ &\quad +\mathrm{rank}(\mathbf{A}_1)\boldsymbol{\Sigma}^{1/2}E[\mathbf{T}_2\mathbf{T}_2']\boldsymbol{\Sigma}^{1/2} \end{aligned}$$

and $E[\mathbf{Q}_{C_3';S_1^{-1}}\boldsymbol{\Sigma}\mathbf{Q}_{C_3';S_1^{-1}}']$ is obtained from [4, (4.2.45)-(4.2.58)]. ■

Theorem 4.6. *For Model II let $\widehat{\boldsymbol{\Sigma}}$ be given in Theorem 2.2, and v_1,v_2,v_3,k_1,k_2, $\mathbf{G}_1$ and $\mathbf{G}_2$ be given in Theorem 4.4. Then,*

$$\begin{aligned} E[n\widehat{\boldsymbol{\Sigma}}] = \boldsymbol{\Sigma}^{1/2}\big\{&(n-\mathrm{rank}(\mathbf{A}_1))\mathbf{I}_p+(\mathrm{rank}(\mathbf{A}_1)-\mathrm{rank}(\mathbf{A}_2))(z_{11}\mathbf{K}_1+\mathbf{P}_{\Sigma^{1/2}G_1}) \\ &+(\mathrm{rank}(\mathbf{A}_2)-\mathrm{rank}(A_3))(z_{12}\mathbf{K}_1+z_{22}\mathbf{K}_2+\mathbf{P}_{\Sigma^{1/2}G_2}) \\ &+\mathrm{rank}(\mathbf{A}_3)(z_{13}\mathbf{K}_1+z_{21}\mathbf{K}_2+z_{33}\mathbf{K}_3+\mathbf{P}_{\Sigma^{1/2}G_3})\big\}\boldsymbol{\Sigma}^{1/2} \end{aligned}$$

where

$$\mathbf{K}_i=\mathbf{P}_{\Sigma^{1/2}G_{i-1}}-\mathbf{P}_{\Sigma^{1/2}G_i},\quad i=1,2,3,\qquad \mathbf{G}_3=\mathbf{G}_2(\mathbf{G}_2'\mathbf{C}_3')^o,$$

$$z_{11}=\tfrac{v_1}{n-\mathrm{rank}(A_1)-v_1-1},\qquad z_{12}=\tfrac{k_1v_2}{n-\mathrm{rank}(A_2)-v_2-1},\qquad z_{22}=\tfrac{v_2}{n-\mathrm{rank}(A_2)-v_2-1},$$

$$z_{13}=\tfrac{k_1k_2v_3}{n-\mathrm{rank}(A_3)-v_3-1},\qquad z_{23}=\tfrac{k_2v_3}{n-\mathrm{rank}(C_3)-v_3-1},\qquad z_{33}=\tfrac{v_3}{n-\mathrm{rank}(A_3)-v_3-1}.$$

Appendix: Proof of Theorem 4.3 (iii)

The proof of the Theorem 4.3 (iii) is very similar to the proof of Theorem 4.3 (ii) and therefore only a few details are given. From Theorem 2.1 it follows that

$$\begin{aligned}\mathbf{K}(\widehat{\mathbf{B}}_1-\mathbf{B}_1)\mathbf{L} &= \mathbf{K}(\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{A}_1'(\mathbf{Y}-E[\mathbf{Y}])\mathbf{S}_3^{-1}\mathbf{C}_1'(\mathbf{C}_1\mathbf{S}_3^{-1}\mathbf{C}_1')^-\mathbf{L} && \text{(A-1)}\\ &-\mathbf{K}(\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{A}_1'\mathbf{A}_2(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1}(\mathbf{Y}-E[\mathbf{Y}])\mathbf{S}_2^{-1}\mathbf{C}_2'(\mathbf{C}_2\mathbf{S}_2^{-1}\mathbf{C}_2')^-\mathbf{L} && \text{(A-2)}\\ &-\mathbf{K}(\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{A}_1'(\mathbf{I}-\mathbf{A}_2(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1})\mathbf{A}_3(\widehat{\mathbf{B}}_3-\mathbf{B}_3)\mathbf{C}_3\mathbf{C}_1'(\mathbf{C}_1\mathbf{C}_1')^-\mathbf{L}. && \text{(A-3)}\end{aligned}$$

Since $\mathbf{A}_1'\mathbf{Y}$ is independent of $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, $\mathbf{Q}_{A_1}\mathbf{Y}$ and $\widehat{\mathbf{B}}_3$, and $\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{Y}$ is independent of $\mathbf{S}_1$, $\mathbf{S}_2$ and $\widehat{\mathbf{B}}_3$ the terms given by (A-1), (A-2), (A-3) are uncorrelated. Thus,

$$\begin{aligned}D[\mathbf{K}\widehat{\mathbf{B}}_1\mathbf{L}] &= D[\mathbf{K}(\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{A}_1'(\mathbf{Y}-E[\mathbf{Y}])\mathbf{S}_3^{-1}\mathbf{C}_1'(\mathbf{C}_1\mathbf{S}_3^{-1}\mathbf{C}_1')^-\mathbf{L}] && \text{(A-4)}\\ &+ D[\mathbf{K}(\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{A}_1'\mathbf{A}_2(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1}(\mathbf{Y}-E[\mathbf{Y}])\mathbf{S}_2^{-1}\mathbf{C}_2'(\mathbf{C}_2\mathbf{S}_2^{-1}\mathbf{C}_2')^-\mathbf{L}] && \text{(A-5)}\\ &+ D[\mathbf{K}(\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{A}_1'(\mathbf{I}-\mathbf{A}_2(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{Q}_{A_1})\mathbf{A}_3(\widehat{\mathbf{B}}_3-\mathbf{B}_3)\mathbf{C}_3\mathbf{C}_1'(\mathbf{C}_1\mathbf{C}_1')^-\mathbf{L}]. && \text{(A-6)}\end{aligned}$$

The dispersion in (A-6) is obtained from Theorem 4.3 (i) and (A-5) can be determined from the treatment of (4.12) via (4.13), i.e (A-5) equals

$$\mathbf{L}'(\mathbf{C}_2\mathbf{C}_2')^-\mathbf{C}_2\mathbf{F}_1\mathbf{C}_2'(\mathbf{C}_2\mathbf{C}_2')^-\mathbf{L}\otimes\mathbf{K}(\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{A}_1'\mathbf{A}_2(\mathbf{A}_2'\mathbf{Q}_{A_1}\mathbf{A}_2)^-\mathbf{A}_2'\mathbf{A}_1(\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{K}', \qquad \text{(A-7)}$$

where $\mathbf{F}_1$ is given by (4.26). Put

$$\mathbf{F}_2 = E[\mathbf{P}_{C_1';S_3^{-1}}\boldsymbol{\Sigma}\mathbf{P}'_{C_1';S_3^{-1}}] \qquad \text{(A-8)}$$

and then (A-4) is determined through

$$\mathbf{L}'(\mathbf{C}_1\mathbf{C}_1')^-\mathbf{C}_1\mathbf{F}_2\mathbf{C}_1(\mathbf{C}_1\mathbf{C}_1')^-\mathbf{L}\otimes\mathbf{K}(\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{K}'.$$

We will copy the approach for obtaining $\mathbf{F}_1$. From (4.15) it follows that

$$\mathbf{F}_2 = \boldsymbol{\Sigma}-\boldsymbol{\Sigma}^{1/2}E[\mathbf{T}_2]\boldsymbol{\Sigma}^{1/2}-\boldsymbol{\Sigma}^{1/2}E[\mathbf{T}_2']\boldsymbol{\Sigma}^{1/2}+\boldsymbol{\Sigma}^{1/2}E[\mathbf{T}_2\mathbf{T}_2']\boldsymbol{\Sigma}^{1/2},$$

where

$$\begin{aligned}\mathbf{T}_2 &= \mathbf{V}_3\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{V}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}, && \text{(A-9)}\\ \mathbf{D}_1^o &= \boldsymbol{\Sigma}^{1/2}\mathbf{C}_1'^o, \qquad \mathbf{V}_3=\mathbf{V}_2+\mathbf{T}_1\mathbf{Z}\mathbf{P}_{Q_{A_1}A_2}\mathbf{Z}'\mathbf{T}_1'.\end{aligned}$$

Moreover

$$\mathbf{D}_1^{o'}\mathbf{V}_3\mathbf{D}_1^o=\mathbf{D}_1^{o'}\mathbf{W}_2\mathbf{D}_1^o+\mathbf{D}_1^{o'}\mathbf{Z}\mathbf{P}_{Q_{A_1}A_2}\mathbf{Z}'\mathbf{D}_1^o\equiv\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o.$$

where $\mathbf{W}_3\sim W_p(\mathbf{I},n-\mathrm{rank}(\mathbf{A}_1))$. Because $\mathcal{R}(\mathbf{D}_1)^\perp\subseteq\mathcal{R}(\mathbf{D}_2)^\perp\subseteq\mathcal{R}(\mathbf{D}_3)^\perp$,

$$\mathbf{D}_2^{o'}\mathbf{V}_3\mathbf{D}_1^o=\mathbf{D}_2^{o'}\mathbf{W}_3\mathbf{D}_1^o, \qquad \mathbf{D}_1^{o'}\mathbf{V}_3\mathbf{D}_1^o=\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o.$$

In correspondence with (4.17) we will study

$$E[\mathbf{T}_2]=\mathbf{P}_{D_2^o}E[\mathbf{T}_2]+\mathbf{P}_{D_3}E[\mathbf{T}_2]+\mathbf{P}_{P_{D_3^o}D_2}E[\mathbf{T}_2],$$

where

$$\mathbf{P}_{P_{D_3^o}D_2}=\mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{D}_3^o)^-\mathbf{D}_3^{o'}\mathbf{D}_2(\mathbf{D}_2'\mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{D}_3^o)^-\mathbf{D}_3^{o'}\mathbf{D}_2)^-\mathbf{D}_2'\mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{D}_3^o)^-\mathbf{D}_3^{o'}$$

is the orthogonal projection on $\mathcal{R}(\mathbf{D}_3)^{\perp} \cap \mathcal{R}(\mathbf{D}_2)$.

It follows that

$$\mathbf{P}_{D_2^o} E[\mathbf{T}_2] = \mathbf{P}_{D_1^o}, \qquad \mathbf{P}_{D_3} E[\mathbf{T}_2] = \mathbf{0}, \qquad \mathbf{P}_{P_{D_3^o} D_2} E[\mathbf{T}_2] = \mathbf{0}$$

which implies

$$E[\mathbf{T}_2] = \mathbf{P}_{D_1^o}.$$

Finally we will consider $E[\mathbf{T}_2\mathbf{T}_2']$. Now

$$\mathbf{P}_{D_2^o} E[\mathbf{T}_2\mathbf{T}_2'] \left(\mathbf{P}_{D_3} : \mathbf{P}_{P_{D_3^o} D_2}\right) = \mathbf{0}, \qquad \mathbf{P}_{D_3} E[\mathbf{T}_2\mathbf{T}_2'] \mathbf{P}_{P_{D_3^o} D_2} = \mathbf{0}$$

and therefore it is enough to separately consider

$$\mathbf{P}_{D_2^o} E[\mathbf{T}_2\mathbf{T}_2'] \mathbf{P}_{D_2^o}, \qquad \mathbf{P}_{D_3} E[\mathbf{T}_2\mathbf{T}_2'] \mathbf{P}_{D_3}, \qquad \mathbf{P}_{P_{D_3^o} D_2} E[\mathbf{T}_2\mathbf{T}_2'] \mathbf{P}_{P_{D_3^o} D_2}.$$

First we observe that

$$\mathbf{P}_{D_2^o} E[\mathbf{T}_2\mathbf{T}_2'] \mathbf{P}_{D_2^o} = E[\mathbf{P}_{D_2^o} \mathbf{W}_3 \mathbf{D}_1^o (\mathbf{D}_1^{o\prime} \mathbf{W}_3 \mathbf{D}_1^o)^- \mathbf{D}_1^{o\prime} \mathbf{D}_1^o (\mathbf{D}_1^{o\prime} \mathbf{W}_3 \mathbf{D}_1^o)^- \mathbf{D}_1^{o\prime} \mathbf{W}_3 \mathbf{P}_{D_2^o}]$$

which because of Wishartness of $\mathbf{W}_3$ equals (see also (4.22))

$$(1-\gamma_4)\mathbf{P}_{D_1^o} + \gamma_4 \mathbf{P}_{D_2^o}, \tag{A-10}$$

where γ_4 was presented in the statement of the theorem. Moreover, we will use that

$$\begin{aligned}
\mathbf{V}_2^{-1} &= \mathbf{V}_2^{-1}\mathbf{D}_2(\mathbf{D}_2'\mathbf{V}_2^{-1}\mathbf{D}_2)^-\mathbf{D}_2'\mathbf{V}_2^{-1} + \mathbf{D}_2^o(\mathbf{D}_2^{o\prime}\mathbf{V}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o\prime}, \\
\mathbf{V}_1^{-1} &= \mathbf{V}_1^{-1}\mathbf{D}_3(\mathbf{D}_3'\mathbf{V}_1^{-1}\mathbf{D}_3)^-\mathbf{D}_3'\mathbf{V}_1^{-1} + \mathbf{D}_3^o(\mathbf{D}_3^{o\prime}\mathbf{V}_1\mathbf{D}_3^o)^-\mathbf{D}_3^{o\prime}
\end{aligned}$$

and then

$$\begin{aligned}
&\mathbf{P}_{D_3} E[\mathbf{T}_2\mathbf{T}_2'] \mathbf{P}_{D_3} \\
&\quad = E[\mathrm{tr}\{\mathbf{V}_3\mathbf{D}_1^o(\mathbf{D}_1^{o\prime}\mathbf{V}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o\prime}\mathbf{D}_1^o(\mathbf{D}_1^{o\prime}\mathbf{V}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o\prime}\mathbf{V}_3\mathbf{V}_1^{-1}\}]\mathbf{P}_{D_3} \\
&\quad = E[\mathrm{tr}\{\mathbf{V}_2\mathbf{V}_2^{-1}\mathbf{V}_3\mathbf{D}_1^o(\mathbf{D}_1^{o\prime}\mathbf{V}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o\prime}\mathbf{D}_1^o(\mathbf{D}_1^{o\prime}\mathbf{V}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o\prime}\mathbf{V}_3\mathbf{V}_2^{-1}\mathbf{V}_2\mathbf{V}_1^{-1}\}]\mathbf{P}_{D_3} \\
&\quad = E[\mathrm{tr}\{\mathbf{V}_2\mathbf{D}_2^o(\mathbf{D}_2^{o\prime}\mathbf{V}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o\prime}\mathbf{V}_3\mathbf{D}_1^o(\mathbf{D}_1^{o\prime}\mathbf{V}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o\prime} \\
&\qquad \times\mathbf{D}_1^o(\mathbf{D}_1^{o\prime}\mathbf{V}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o\prime}\mathbf{V}_3\mathbf{D}_2^o(\mathbf{D}_2^{o\prime}\mathbf{V}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o\prime}\mathbf{V}_2\mathbf{V}_1^{-1}\}]\mathbf{P}_{D_3} \\
&\quad = E[\mathrm{tr}\{\mathbf{D}_3^{o\prime}\mathbf{W}_2\mathbf{D}_2^o(\mathbf{D}_2^{o\prime}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o\prime}\mathbf{W}_3\mathbf{D}_1^o(\mathbf{D}_1^{o\prime}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o\prime} \\
&\qquad \times\mathbf{D}_1^o(\mathbf{D}_1^{o\prime}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o\prime}\mathbf{W}_3\mathbf{D}_2^o(\mathbf{D}_2^{o\prime}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o\prime}\mathbf{W}_2\mathbf{D}_3^o(\mathbf{D}_3^{o\prime}\mathbf{V}_1\mathbf{D}_3^o)^-\}]\mathbf{P}_{D_3}.
\end{aligned} \tag{A-11}$$

By assumption there exist matrices $\mathbf{U}_1$ and $\mathbf{U}_2$ so that

$$\mathbf{D}_1^o = \mathbf{D}_2^o\mathbf{U}_1, \qquad \mathbf{D}_2^o = \mathbf{D}_3^o\mathbf{U}_2.$$

Hence, (A-11) equals

$$\begin{aligned}
&E[\mathrm{tr}\{(\mathbf{D}_2^{o\prime}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o\prime}\mathbf{W}_3\mathbf{D}_1^o(\mathbf{D}_1^{o\prime}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o\prime} \\
&\qquad \times\mathbf{D}_1^o(\mathbf{D}_1^{o\prime}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o\prime}\mathbf{W}_3\mathbf{D}_2^o(\mathbf{D}_2^{o\prime}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{U}_2'\mathbf{D}_3^{o\prime}\mathbf{W}_2 \\
&\qquad \times\mathbf{D}_3^o(\mathbf{D}_3^{o\prime}\mathbf{V}_1\mathbf{D}_3^o)^-\mathbf{D}_3^{o\prime}\mathbf{W}_2\mathbf{D}_3^o\mathbf{U}_2\}]\mathbf{P}_{D_3}.
\end{aligned} \tag{A-12}$$

Furthermore, according to Lemma 4.1

$$\mathbf{N}_1 \;=\; (\mathbf{D}_3^{o'}\mathbf{W}_2\mathbf{D}_3^o)^{-1/2}\mathbf{D}_3^{o'}\mathbf{V}_1\mathbf{D}_3^o(\mathbf{D}_3^{o'}\mathbf{W}_2\mathbf{D}_3^o)^{-1/2},$$

is independent of $\mathbf{D}_3^{o'}\mathbf{W}_2\mathbf{D}_3^o$ and $\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o$,

$$E[\mathbf{N}_1^{-1}] = \gamma_5\mathbf{I}_{p-\mathrm{rank}(C_3)}.$$

Similarly,

$$\mathbf{N}_2 \;=\; (\mathbf{D}_2^{o'}\mathbf{W}_3\mathbf{D}_2^o)^{-1/2}\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_3\mathbf{D}_2^o)^{-1/2}$$

is independent of $\mathbf{D}_2^{o'}\mathbf{W}_3\mathbf{D}_2^o$,

$$E[\mathbf{N}_2^{-1}] = \gamma_6\mathbf{I}_{p-\mathrm{rank}(C_2)},$$

$$\mathbf{N}_3 \;=\; (\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^{-1/2}\mathbf{D}_1^{o'}\mathbf{W}_2\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^{-1/2}$$

is independent of $\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o$,

$$\begin{aligned} E[\mathbf{N}_3] &= \gamma_7\mathbf{I}_{p-\mathrm{rank}(C_1)},\\ E[\mathbf{N}_3^{-1}] &= \gamma_8\mathbf{I}_{p-\mathrm{rank}(C_1)}. \end{aligned}$$

Then, (A-12) equals

$$\begin{aligned} &E[\mathrm{tr}\{(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}\mathbf{W}_3\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\\ &\quad\times\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{U}_2'(\mathbf{D}_3^{o'}\mathbf{W}_2\mathbf{D}_3^o)^{1/2}\mathbf{N}_1^{-1}(\mathbf{D}_3^{o'}\mathbf{W}_2\mathbf{D}_3^o)^{1/2}\mathbf{U}_2\}]\mathbf{P}_{D_3}\\ &= \gamma_5 E[\mathrm{tr}\{\mathbf{D}_2^{o'}\mathbf{W}_3\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\}]\mathbf{P}_{D_3}\\ &= \gamma_5 E[\mathrm{tr}\{(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{U}_1'(\mathbf{D}_2^{o'}\mathbf{W}_3\mathbf{D}_2^o)^{1/2}\mathbf{N}_2^{-1}(\mathbf{D}_2^{o'}\mathbf{W}_3\mathbf{D}_2^o)^{1/2}\mathbf{U}_1\}]\mathbf{P}_{D_3}\\ &= \gamma_5\gamma_6 E[\mathrm{tr}\{(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o\}]\mathbf{P}_{D_3}\\ &= \gamma_5\gamma_6 E[\mathrm{tr}\{\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\}]\mathbf{P}_{D_3} = \gamma_4\gamma_5\gamma_6\mathbf{P}_{D_3}. \end{aligned} \tag{A-13}$$

The last expression which will be considered requires also some calculations:

$$\begin{aligned} &\mathbf{P}_{P_{D_3^o}D_2}E[\mathbf{T}_2\mathbf{T}_2']\mathbf{P}_{P_{D_3^o}D_2}\\ &\quad= E[\mathbf{P}_{P_{D_3^o}D_2}\mathbf{V}_3\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{V}_3\mathbf{P}_{P_{D_3^o}D_2}]\\ &\quad= E[\mathbf{P}_{P_{D_3^o}D_2}\mathbf{V}_2\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{V}_2\mathbf{P}_{P_{D_3^o}D_2}] &&\text{(A-14)}\\ &\qquad+E[\mathbf{P}_{P_{D_3^o}D_2}\mathbf{V}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}(\mathbf{W}_3-\mathbf{W}_2)\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\\ &\qquad\quad\times\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{V}_2\mathbf{P}_{P_{D_3^o}D_2}] &&\text{(A-15)}\\ &\qquad+E[\mathbf{P}_{P_{D_3^o}D_2}\mathbf{V}_2\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}(\mathbf{W}_3-\mathbf{W}_2)\\ &\qquad\quad\times\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{P}_{P_{D_3^o}D_2}] &&\text{(A-16)}\\ &\qquad+E[\mathbf{P}_{P_{D_3^o}D_2}\mathbf{V}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}(\mathbf{W}_3-\mathbf{W}_2)\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\\ &\qquad\quad\times\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}(\mathbf{W}_3-\mathbf{W}_2)\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}\mathbf{V}_2\mathbf{P}_{P_{D_3^o}D_2}]. &&\text{(A-17)} \end{aligned}$$

Now (A-14) equals

$$E[\mathrm{tr}\{\mathbf{W}_2\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\}]\mathbf{P}_{P_{D_3^o}D_2} = \gamma_4\gamma_7\mathbf{P}_{P_{D_3^o}D_2}. \tag{A-18}$$

Turning to (A-15) this expression equals

$$E[\mathbf{P}_{P_{D_3^o}D_2}\mathbf{W}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}(\mathbf{W}_3-\mathbf{W}_2)\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{W}_2\mathbf{P}_{P_{D_3^o}D_2}]$$
$$= E[\mathrm{tr}\{(\mathbf{W}_3-\mathbf{W}_2)\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\}]\mathbf{P}_{P_{D_3^o}D_2}$$
$$= E[\mathrm{tr}\{\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\} - \mathrm{tr}\{\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{W}_2\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\}]\mathbf{P}_{P_{D_3^o}D_2}$$
$$= \gamma_4(1-\gamma_7)\mathbf{P}_{P_{D_3^o}D_2}. \tag{A-19}$$

By symmetry we obtain the same expression for (A-16). Finally it is observed that (A-17) equals

$$E[\mathbf{P}_{P_{D_3^o}D_2}\mathbf{W}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}(\mathbf{W}_3-\mathbf{W}_2)\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}(\mathbf{W}_3-\mathbf{W}_2)$$
$$\times\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{P}_{P_{D_3^o}D_2}]$$
$$= E[\mathrm{tr}\{\mathbf{W}_2\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}(\mathbf{W}_3-\mathbf{W}_2)\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}$$
$$\times\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}(\mathbf{W}_3-\mathbf{W}_2)\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}\}]\mathbf{P}_{P_{D_3^o}D_2}$$
$$= E[\mathrm{tr}\{\mathbf{W}_3\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_2^o(\mathbf{D}_2^{o'}\mathbf{W}_2\mathbf{D}_2^o)^-\mathbf{D}_2^{o'}\}]\mathbf{P}_{P_{D_3^o}D_2}$$
$$-2E[\mathrm{tr}\{\mathbf{W}_3\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\}]\mathbf{P}_{P_{D_3^o}D_2}$$
$$+E[\mathrm{tr}\{\mathbf{W}_2\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\}]\mathbf{P}_{P_{D_3^o}D_2}$$
$$= E[\mathrm{tr}\{(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{U}_1'(\mathbf{D}_2^{o'}\mathbf{W}_3\mathbf{D}_2^o)^{1/2}\mathbf{N}_2^{-1}(\mathbf{D}_2^{o'}\mathbf{W}_3\mathbf{D}_2^o)^{1/2}\mathbf{U}_1\}]\mathbf{P}_{P_{D_3^o}D_2}$$
$$-2E[\mathrm{tr}\{\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\}]\mathbf{P}_{P_{D_3^o}D_2}$$
$$+E[\mathrm{tr}\{(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^{-1/2}\mathbf{D}_1^{o'}\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^{-1/2}\mathbf{N}_3\}]\mathbf{P}_{P_{D_3^o}D_2}$$
$$= (\gamma_6-2+\gamma_7)E[\mathrm{tr}\{\mathbf{D}_1^o(\mathbf{D}_1^{o'}\mathbf{W}_3\mathbf{D}_1^o)^-\mathbf{D}_1^{o'}\}]\mathbf{P}_{P_{D_3^o}D_2} = \gamma_4(\gamma_6-2+\gamma_7)\mathbf{P}_{P_{D_3^o}D_2}. \tag{A-20}$$

Thus, summing (A-10), (A-13), (A-18), (A-19) and (A-20) we obtain

$$E[\mathbf{T}_2\mathbf{T}_2'] = (1-\gamma_4)\mathbf{P}_{D_1^o} + \gamma_4\mathbf{P}_{D_2^o} + \gamma_4\gamma_5\gamma_6\mathbf{P}_{D_3} + \gamma_4\gamma_6\mathbf{P}_{P_{D_3^o}D_2}$$

and then $\mathbf{F}_2$ given in the statement of the theorem is obtained.

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